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ON-LINE PROCESS MODEL UPDATE IN DISCRETE-TIME PREDICTIVE CONTROLLERS: A ROBUST APPROACH

Guilherme Augusto de Almeida Gonçalves

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Orientadores: Argimiro Resende Secchi
Maurício Bezerra de Souza Jr.
Martin Guay

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Examinada por:

Prof. Argimiro Resende Secchi, D.Sc.

Prof. Maurício Bezerra de Souza Jr., D.Sc.

Prof. Príamo Albuquerque Melo Jr., D.Sc.

Prof. Darci Odloak, Ph.D.

Eng. Mario Cesar Mello Massa de Campos, D.Sc.

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Maurício Bezerra de Souza Jr.
Martin Guay

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"Na vida só vale o amor e a amizade. O resto é tudo pinóia, é tudo presunção, não paga a pena."

Jorge Amado

Para minha mãe Soraia, pelo
amor, encorajamento e exemplo.
"Feliz o homem capaz de ter por
alguém tanto respeito que a
simples lembrança do modelo
basta para lhe dar ordem e
harmonia espiritual"
Sêneca

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Dorival Caymmi

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ATUALIZAÇÃO ONLINE DE MODELOS DE PROCESSO EM CONTROLADORES PREDITIVOS DISCRETOS: UMA ABORDAGEM ROBUSTA

Guilherme Augusto de Almeida Gonçalves

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Orientadores: Argimiro Resende Secchi
Maurício Bezerra de Souza Jr.
Martin Guay

Programa: Engenharia Química

A perda de desempenho de um sistema de controle pode ser causada por diversos fatores. Em controladores preditivos, um dos aspectos relevantes para o sucesso da estratégia é a utilização de um modelo que possa efetuar as predições de maneira coerente com a planta. Entretanto, durante a campanha da planta, é usual a perda de performance do sistema de controle, muitas vezes causada por mudanças de pontos de operação em processos intrinsecamente não lineares ou por característica natural de processos químicos, que são variantes no tempo, como reações com desativação de catalisadores e trocadores de calor, que sofrem com mudanças em seus coeficientes de troca térmica. Nessa tese, analisaram-se diversos algoritmos para a atualização de modelos em controladores preditivos. Primeiro, foram avaliados os algoritmos clássicos, mostrando a possibilidade de melhoria de desempenho utilizando a estratégia de atualização. A seguir, um algortimo de controle baseado em estimação de parâmetros por intervalos e realimentação de estados foi utlizado para propor um esquema de controle preditivo robusto. Por fim, um algoritmo de estimação simultânea de parâmetros e estados foi elaborado para sistemas discretos e aplicado em um esquema de controle preditivo com atualização de parâmetros, apresentando desempenho satisfatório.

Abstract of Thesis presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Doctor of Science (D.Sc.)

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Guilherme Augusto de Almeida Gonçalves

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Advisors: Argimiro Resende Secchi
Maurício Bezerra de Souza Jr.
Martin Guay

Department: Chemical Engineering

The performance degradation of a control system may be related to several factors. In predictive controllers, one relevant aspect for its success is to use a model that can perform consistent plant predictions. However, after the control system commissioning, a performance degradation may occur. This degradation can be caused, for example, by a new operating point in a nonlinear system or by a time varying chemical process, such as, a reactor with a catalyst deactivation or a heat exchanger with a heat transfer coefficient that changes along the operation. The updating of those models is a critical task and may lead to instability if it is not properly conducted. In this thesis, several algorithms for model update in predictive controllers were analyzed. In this sense, an in-depth study on the use of classical parameter and state estimators was performed, evaluating the gains of these strategies in predictive controllers. Moreover, an algorithm was proposed to update the model when the state vector can be measured completely based on interval observers, resulting in a robust MPC. We then proceeded to an algorithm to update the models dispensing the complete state vector measurement requirement. Therefore, an algorithm was developed for joint estimation of states, parameters and deterministic uncertainty region. Finally, this algorithm was applied and tested for multivariable predictive control.

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List of Symbols

A_R	Thermal exchange area, p. 36
B	Uncertainty ball from the state-feedback MPC approach, p. 58
C_A	A concentration, p. 36
C_B	B concentration, p. 36
C_p	Reactor wall conductivity, p. 36
F	Reactor feed, p. 36
$G(x_k, u_k)$	Nonlinear function from the state-feedback MPC approach, p. 59
K_k	Correction factor from the state-feedback MPC approach, p. 59
M_v	Disturbance upper bound from the estimation and output-feedback problem, p. 93
N	Control horizon length, p. 18
Q	State covariance matrix, p. 13
R	Measurement covariance matrix, p. 13
T	Reactor temperature, p. 36
T_0	Feed temperature, p. 36
T_K	Jacket temperature, p. 36
V	Cost Function, p. 18
V	Reactor Volume, p. 36
$V(k)$	Lyapunov function candidate from the state and parameter estimation problem, p. 95

W	Terminal Penalty, p. 64
W	*Unscented* Filter Weights, p. 13
X_f	Terminal set, p. 18
$\mathbf{B}$	Nonlinear function from the estimation and output-feedback problem, p. 93
$\mathbf{C}(k)$	Filter from the estimation and output-feedback problem, p. 93
$\Delta H_{()}$	Reaction enthalpy, p. 36
Γ	UKF tuning parameter, p. 13
$\Sigma(k)$	Identification matrix from the parameter estimation problem, p. 98
Θ	CEKF decision variables, p. 12
Θ_k	Constraint set ball of the parameters from the state-feedback MPC approach, p. 60
$\mathbb{X}_f$	Terminal Constraint set, p. 64
α_i	Parameter from the chemotherapy model, p. 84
β_i	Parameter from the chemotherapy model, p. 84
$\boldsymbol{\kappa_N(x)}$	MPC control law, p. 18
χ	Sigma Points, p. 13
$\mathbf{d}$	Disturbance vector, p. 31
ℓ_d	Output to disturbance map, p. 31
ℓ_x	Output to state map, p. 31
$\mathbf{e}$	Deviation between the disturbed system and the nominal system, p. 25
η_k	Auxiliary variable from the state-feedback MPC approach, p. 59
$\boldsymbol{\eta}(k)$	Auxiliary variable from the state and parameter estimation problem, p. 94
γ_i	Parameter from the chemotherapy model, p. 84

$\hat{\theta}_k$	Vector of parameter estimates from the state-feedback MPC approach, p. 59
$\mathbf{F}_{k-1}$	Partial derivatives matrix with respect to the states, p. 9
$\mathbf{H}_k$	Partial derivatives matrix with respect to the measurements, p. 9
$\mathbf{I}$	Identity, p. 39
$\mathbf{K}$	Filter Gain, p. 10
$\mathbf{L}_{k-1}$	Partial derivatives matrix with respect to the model noise, p. 9
$\mathbf{M}_k$	Partial derivatives matrix with respect to the measurement noise, p. 9
$\mathbf{P}$	Error Covariance Matrix, p. 10
$\mathbb{R}$	Real numbers set, p. 18
$\mathbb{U}$	Input set, p. 18
$\mathbb{X}$	State set, p. 18
ω	Model noise, p. 8
ω_k	Filter variable from the state-feedback MPC approach, p. 59
θ_k	Parameter from the state-feedback MPC approach, p. 58
θ	Recursively estimated parameter, p. 39
$\mathbf{u}^0(x)$	Optimal solution, p. 18
$\mathbf{u}_k$	Input vector of a dynamic discrete-time system, p. 18
ϑ_k	Disturbance from the state-feedback MPC approach, p. 58
$\mathbf{v}(k)$	Output disturbance from the estimation and output-feedback problem, p. 93
$\mathbf{w}(k)$	State disturbance from the estimation and output-feedback problem, p. 93
$\mathbf{x}(k)$	State from the estimation and output-feedback problem, p. 93
$\mathbf{x}_k$	State of a dynamic discrete-time system, p. 18

$\mathbf{y}(k)$	Output from the estimation and output-feedback problem, p. 93
$\mathbf{z}$	Nominal system state in the tube approach, p. 25
e_k	State estimation error at time step k from the state-feedback MPC approach, p. 59
f	System equations, p. 9
f_a	Dynamic system including disturbances, p. 31
f_a	Measurement equation including disturbances, p. 31
h	Measurement equations, p. 9
$k_{()}$	Kinetics constants, p. 36
r_i	Parameter from the chemotherapy model, p. 84
u_k	Control action from the state-feedback MPC approach, p. 58
v	Measurement Noise, p. 8
x_k	State from the state-feedback MPC approach, p. 58
x_k	System state, p. 8
y_k	Measurement, p. 8
z_θ	Uncertainty radius from the state-feedback MPC approach, p. 58

List of Abbreviations

ANMPC	*Adaptive Nonlinear Model Predictive Control*, p. 51
ARMAX	*AutoRegressive Moving Average with eXogenous input*, p. 5
ARX	*AutoRegressive with eXogenous input*, p. 7
CEKF	*Constrained Extended Kalman Filter*, p. 8
CLF	*Control Lyapunov Function*, p. 21, 64
CSTR	*Continuous Stirred-Tank Reactor*, p. 36
CUKF	*Constrained Unscented Kalman Filter*, p. 16
DMC	*Dynamic Matrix Control*, p. 51
EKF	*Extended Kalman Filter*, p. 8
GPC	*Generalized Predictive Controller*, p. 6
JILT	*Just-in-Time Learning*, p. 6
MHE	*Moving Horizon Estimation*, p. 8
MPC	*Model Predictive Control*, p. 1, 4
NARMAX	*Nonlinear AutoRegressive Moving Average with eXogenous input*, p. 4
NMPC	*Nonlinear Model Predcitive Control*, p. 18
PI	*Proporcional-Integral Controller*, p. 5
PRBS	*Pseudo Random Binary Sequence*, p. 4
QP	*Quadratic Programming*, p. 43
RLS	*Recursive Least Squares*, p. 4
UKF	*Unscented Kalman Filter*, p. 13

Chapter 1

Introduction

The idea of using closed-loop process data to modify the controller parameters is not new. The adaptive control techniques, such as self-tuning, apply recursive estimation principles for control law updating. According to KANO and OGAWA (2010), such systems have some acceptance and industrial use. Moreover, OGUNNAIKE and RAY (1994) enumerate two essential aspects for successfully apply this strategy: the parameter estimation algorithm and the use of procedures to obtain an adequate experimental planning. This approach, using linear controllers in SISO (single-input, single-output) systems, was shown by ASTRÖM and WITTENMARK (1973) in the 70's.

Although it is not a new strategy, a practical approach for adaptive control in chemical process industry is scarce. A survey conducted in the Japanese industry, which results are shown in KANO and OGAWA (2010), shows that in nineteen industries consulted, only two facilities use an adaptive control algorithm and none of them has a standardized implementation procedure. However, model predictive control (MPC) algorithm has wide application, being present in sixteen out of nineteen industries. This algorithm has been widely accepted because of its inherent ability to handle processes with long transients, strict quality requirements, easy implementation of constraints, among other advantages (DONES *et al.*, 2010). The predictive control algorithm based on linear models is a widespread industrial process control strategy, mostly used in distillation columns (KANO and OGAWA, 2010).

The successful predictive control application is directly related to the model predictive ability. For example, to obtain an empirical model for a distillation column and its effective implementation are required around fifteen days (KANO and OGAWA, 2010). Assuming the process operation may be subject to inherent changes due to campaign time, maybe this procedure to obtain the model will be repeated during this period, or the MPC layer will be turned off by the operation due to poor performance.

In the survey carried out by KANO and OGAWA (2010), industries were asked which were the main desired improvements in MPC technology, the results are shown in Table 1.1

Table 1.1: Requirements reported by the chemical industries to improve MPC theory KANO and OGAWA (2010).

Requirement	Percentage
To cope with changes in process characteristics	26%
To develop relationships between model accuracy and control performance	24%
To cope with unsteady operation	16%
Use of know-how in control system	16%
To cope with nonlinearity	13%
Others	5%

It is noticeable, among the main industrial demands, the requirement to make the MPC technology capable of operating in systems where changes occur and, consequently, controller performance is reduced. In the same work, it was asked which alternatives would be more suitable for industrial use. The results are available in Table 1.2.

Table 1.2: Need for improvement in response to changes and nonlinearity KANO and OGAWA (2010).

Requirement	Percentage
Switch multiple linear models	28%
To improve linear MPC robustness	25%
To use time-varying or nonlinear models	18%
Adaptive function for linear MPC	18%
Others	11%

Data such as those presented by KANO and OGAWA (2010) are difficult to find for the remaining worldwide industrial facilities. However, these data show that there are gaps in the present technology and theory, which can be seen as an opportunity for improvement, in terms of performance loss and control system maintenance.

This work proposes some strategies to cope with online model updating in MPC. We have employed well-known strategies, such as the extended Kalman Filter, in order to update the models and apply an output-feedback strategy, the results showed that this scheme may lead to a solid performance improvement. However, the online uncertainty estimation using Kalman Filters is still a gap in the literature. In order to comply with the parameter uncertainty and then use a robust MPC, we presented an alternative, based on interval observers approach, for the state feedback robust nonlinear MPC under uncertain parameters. Finally, since in many industrial problems full state measurement is not available, we have formulated the extension of the set-based approach for joint estimation in a discrete-time framework, which results in an output-feedback robust MPC strategy for a class of nonlinear systems. Furthermore, this robust MPC can be solved online by explicitly using Lipschitz constants, avoiding the unsolvable nonlinear min-max optimization and resulting in the same computational burden as the nominal MPC.

The remainder of this thesis is organized as follows: in Chapter 2, the state-of-art of adaptive MPC and state estimation is explored, the current approaches for MPC and parameter estimation stability are carefully discussed. Then, the current technologies for model updating are analyzed, most of them using Kalman Filters or recursive least squares (RLS) for parameter and state-estimation. In Chapter 3, it is showed that this approach may result in robustness improvement, despite the difficulties to estimate and control the error prediction. In Chapter 4, a strategy for nonlinear discrete-time state-feedback MPC and parameter update that guarantees robust stability is presented. Since, in many chemical processes, full state measurement is not available, a way for state and parameter estimation based on interval observers approach is proposed in 5.2. Additionally, in Chapter 5, the proposed interval observer is applied in an output-feedback MPC framework. Finally, in Chapter 6, the final comments and future research suggestions are presented.

Chapter 2

Theory and literature review

In this chapter we present a broad review about the thesis topics and the state of art of model predictive control, focusing on model updating and state and parameter estimation. In the remainder chapters, a more specific literature review is given in the introduction section of each chapter.

2.1 MPC coupled with parameter estimation

Since the early years of MPC, there have been attempts to develop linear controller extensions for application in nonlinear or time variants plants. In the first works, some effort was made to obtain a controller that could preserve the mathematical complexity of linear MPC and handle nonlinear systems. Among these works, it is the controller proposed by MORNINGRED *et al.* (1992), which makes use of empirical NARMAX models. These models, when using a linear parameter structure, are represented by the Equation 2.1 (using the same nomenclature adopted by MORNINGRED *et al.* (1992)):

$$\mathbf{y}(t) = \boldsymbol{\theta}^T(t-1)\boldsymbol{\phi}(t-1) + \boldsymbol{\epsilon}(t) \tag{2.1}$$

where $\boldsymbol{\theta}(t-1)$ is the parameter vector estimated at $t-1$, $\boldsymbol{\phi}(t-1)$ is a past values function vector (regressor), $\mathbf{y}(t)$ the model output and $\boldsymbol{\epsilon}(t)$ an output disturbance. In this paper, an estimator of the least squares recursive (RLS) type was used to fit the vector $\boldsymbol{\theta}(t-1)$ to the process conditions subject to a PRBS (Pseudo Random Binary Sequence) signal. In this algorithm, the parameter vector was updated as follows:

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1) + \mathbf{K}(t)[\mathbf{y}(t) - \boldsymbol{\phi}^T(t)\boldsymbol{\theta}(t-1)] \tag{2.2}$$

The estimation Gain $\mathbf{K}(t)$ is updated as:

$$\mathbf{K}(t) = \mathbf{P}(t-1)\boldsymbol{\phi}(t)[\lambda + \boldsymbol{\phi}^T(t)\mathbf{P}(t-1)\boldsymbol{\theta}(t)]^{-1} \tag{2.3}$$

The covariance matrix ($\mathbf{P}(t)$) is given by:

$$\mathbf{P}(t) = \frac{\mathbf{P}(t-1)}{\lambda}[\mathbf{I} - \mathbf{K}(t)\boldsymbol{\phi}^T(t)] \tag{2.4}$$

in which λ is the forgetting factor.

In this work, a CSTR was used to show that this algorithm can result in a unstable system if the controller does not handle plant/model mismatch. It should be noted that in the 90's few papers on the study of MPC controller stability had been proposed and many of the robust predictive control strategies were developed in the subsequent years. Despite the theoretical limitations, in the following year, the authors presented an application to an experimental distillation column, showing that the performance is superior to the linear MPC and the PI controllers using decoupling (MORNINGRED *et al.*, 1993).

Before the MORNINGRED *et al.* (1992) publication, other authors had used the adaptive predictive strategy, for example, the controller proposed by SOUZA JR. (1989), which uses the GPC (Generalized Predictive Control) in combination with a RLS estimator was applied to a distillation column control.

Other authors have employed variations of this strategy for a broad range of chemical processes. DEFAYE *et al.* (1993) applied the RLS-MPC for batch reactors control using linear models, and showed that some problems may arise when this approach is used. One of the main issues is related to the estimator ability to perceive small process variations and, at the same time, to be insensitive to noise, this property may have a large influence in the controller performance.

CLARKE (1996) proposed a method similar to the DEFAYE *et al.* (1993), however, elements for guarantee stability were used, such as terminal constraints. Moreover, an algorithm was used for choosing the model order, given a specified maximum order.

RHO *et al.* (1988) proposed a strategy for a polymerization reactor control by using online identified ARMAX models. After model identification, it was used in a MPC for the jacket temperature setpoint calculation, which was used in cascade with a PI controller. Additionally, by a normalization procedure to ensure that the disturbances were finite, the recursive least squares technique become less susceptible to failure. Furthermore, a dead zone for gain estimation was chosen as follows:

$$\mathbf{K}(t) = \begin{cases} \mathbf{K} \in \left(0, \ \frac{\mu}{\mu+1}\right) \quad if \quad |\epsilon(t)| \geq (1+\mu)^{1/2}\Omega \\ \mathbf{0}, \quad otherwise \end{cases} \tag{2.5}$$

where μ e Ω are the tuning parameters. In that work, an experimental validation was done, showing a superior perfomance in comparison with a PID controller.

SECCHI *et al.* (2001) have compared the predictive controller with RLS parameter updating with other strategies such as the nonlinear predictive controller (NMPC) and a local models network. It was observed that approximate methods can find a solution to the control problem close to the result obtained by the NMPC. KARER *et al.* (2008) extended the methods for predictive controller using RLS and dead zone for hybrid systems[1] control, using a batch reactor as a case study.

Through the literature analysis until the 90's, it is noticed that a few works attempted to combine the robust control techniques and parameters estimation in a single algorithm. In subsequent years, the first publications using strategies to explicitly take into account the uncertainty measurements and parameter estimates began to emerge along with the evolution of robust control theory. In the work of FUKUSHIMA *et al.* (2007), a parameter estimation routine combined with a robust predictive controller was used. The authors emphasized some issues, such as the constraints attendance under the adaptive algorithm, the necessity to predict the future parameters behavior in order to satisfy the constraints and, finally, the challenge of theoretical feasibility and stability. FUKUSHIMA *et al.* (2007) proposed the following problem:

The parameter error is given by:

$$\tilde{\theta} = \theta - \theta^* \tag{2.6}$$

where θ^* denotes the uncertain parameter vector and θ the true values. Once the future value of $||\tilde{\theta}(t)||$ is unknown, a bound $\nu(t|t_i)$ can be estimated. In this work it was proposed an algorithm for linear systems that satisfies an exponential decay:

$$\nu(t|t_i) = \nu(t_i)e^{-K\gamma(t_i)(t-t_i)} \tag{2.7}$$

where K is the feedback gain and γ a parameter used in the estimation algorithm. Using this approach, a less conservative update for the parametric uncertainty can be obtained.

Other works have tried to improve the estimation provided by the RLS algorithm,

[1] hybrid systems are systems that involve interaction between discrete and continuous dynamics

which can generate poor parameters estimates if a degree of excitation is not reached. In the work of KANSHA and CHIU (2009), the Just-in-Time Learning (JITL) technique was employed, using a GPC controller. In the JILT strategy, a data set is initially built in an open loop and, as the process progresses, it is updated. In this approach only the most relevant data are selected. In general, the criterion used is given by (given two sets x_q and x_i):

$$s_i = \kappa\sqrt{e^{-||x_q - x_i||}} + (1-\kappa)cos(\theta_i) \tag{2.8}$$

where $0 < \kappa \leq 1$ and θ is the angle between the sets Δx_q and Δx_i. The parameter s_i is a similarity measurement, which is close to 1 when the two sets are similar. In general, the data set is used for a linear low order model estimation. In KANSHA and CHIU (2009), a second order ARX model was identified. This strategy showed a superior performance in comparison with the MPC-RLS. It improves the model only when it is necessary, ceasing as soon as the steady state is achieved.

Theoretical studies about the adaptive MPC stability were shown in the work of ADETOLA *et al.* (2009). The conditions for guaranteeing stability under parameter adaptation and close-loop MPC were outlined. The first step was a parameter estimation algorithm development, in which the uncertainty bounds were estimated. Moreover, the routine guarantees the uncertainty set updating only when this set contracts. This parameter estimation was used along with a robust MPC under the min-max formulation. Since the uncertainty set is updated along time, the control actions are less conservative than the usual robust MPC. In this approach an excitation constraint must be satisfied by the optimization routine. The estimation algorithm is given by:

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}(\mathbf{C} - \mathbf{Q}\hat{\boldsymbol{\theta}}), \quad \hat{\boldsymbol{\theta}}(t_0) = \boldsymbol{\theta}^0 \tag{2.9}$$

where Γ is the filter gain, $\hat{\boldsymbol{\theta}}$ the parameter estimates and $\mathbf{Q}$, $\mathbf{C}$ are given by the differential equation:

$$\dot{\mathbf{Q}} = \mathbf{w}^T\mathbf{w}, \qquad \mathbf{Q}(t_0) = 0 \tag{2.10}$$

$$\dot{\mathbf{C}} = \mathbf{w}^T(\mathbf{w}\boldsymbol{\theta}^0 + \mathbf{x} - \hat{\mathbf{x}} - \boldsymbol{\eta}), \quad \mathbf{C}(t_0) = 0 \tag{2.11}$$

$$\dot{\boldsymbol{\eta}} = -k_w\boldsymbol{\eta}, \quad \boldsymbol{\eta}(t_0) = \mathbf{e}(t_0) \tag{2.12}$$

$\mathbf{w}$ is a first order filter and $\boldsymbol{\eta}$ an auxiliary variable, in which k_w is a gain that must be tuned. ADETOLA *et al.* (2009) show that by using this algorithm, the error upper bound is given by:

$$||\tilde{\boldsymbol{\theta}}(t)|| \leq exp^{-\epsilon(t_c)(t-t_0)}||\tilde{\boldsymbol{\theta}}(t_0)||, \ \forall t \geq t_c \tag{2.13}$$

where ($||\tilde{\boldsymbol{\theta}}|| = ||\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}||$) and $\epsilon = \lambda_{min}(\Gamma \mathbf{Q}(t))$ is the lowest eigenvalue. It is possible to show that this algorithm has a finite time convergence. The robust stability details can be found in ADETOLA and GUAY (2011).

In recent years many algorithms have been revisited with improvements. AUMI and MHASKAR (2013) revisited the RLS algorithm for model update. In this proposal, called probabilistic RLS, the probability of the process model being representative is taken into account for the estimation, thus unnecessary updates are avoided. Moreover, the rate of model updating can be tuned.

There are other ways for model updating, among these, the most common is the use of neural networks with closed loop estimation to perform the predictive control with adaptation. In the work of AKPAN and HASSAPIS (2011) a RLS algorithm was used for neural networks online estimation, which were applied in two strategies, the first using linear control based on linearization and application of the GPC control. The second based on direct use of the neural networks in a NMPC strategy. SALAHSHOOR *et al.* (2013) employed the neural network for a gas-lift process control, the network were updated using the Unscented (UKF) and the Extended (EKF) Kalman Filters.

Recently, CHAN *et al.* (2014) proposed a strategy for discontinuous model updating in a model predictive controller. This approach is able to detect the obsolete transfer functions when the predictions of the control model deviate from disturbance estimate by more than a pre-determined amount. Moreover, the algorithm detects the necessary excitation level for the parameter estimation.

2.2 State estimators

The state of a dynamic system is defined as the smallest set of variables such that the knowledge of these at the initial time (t_0) and the system inputs at later times completely determines the system behavior at any time-step (OGATA, 2010). Thus, the state is uniquely determined by the state at initial time and the input for $t \geq t_0$.

If it is not possible to measure the full system state, or even if the system output consists of linear or non-linear state combinations, it is necessary the use of state estimators (MACIEJOWSKY, 2000). For non-linear systems, some techniques are used, such as: the extended Kalman filter (EKF), the constrained Kalman filter (CEKF), the moving horizon estimators (MHE), and the Unscented Kalman Filter.

Each one of these strategies is shown below.

2.2.1 Extended Kalman Filter

Consider a nonlinear discrete-time system:

$$\begin{aligned}
x_k &= f(x_{k-1}, u_{k-1}, \omega_{k-1}) \\
y_k &= h(x_k, \nu_k) \\
\omega &\sim (0, Q_k) \\
\nu &\sim (0, R_k)
\end{aligned} \tag{2.14}$$

In Equation (2.14), h and f are nonlinear functions. A first-order Taylor series expansion of these functions around $\hat{x}_{k-1}^{+}$ are given in Equation (2.15) (SIMON, 2006):

$$\begin{aligned}
x_k &= f(\hat{x}_{k-1}^{+}, u_{k-1}, 0) + \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1}^{+}} (x_{k-1} - \hat{x}_{k-1}^{+}) + \left.\frac{\partial f}{\partial \omega}\right|_{\hat{x}_{k-1}^{+}} \omega_{k-1} \\
&= f(\hat{x}_{k-1}^{+}, u_{k-1}, 0) + F_{k-1}(x_{k-1} - \hat{x}_{k-1}^{+}) + L_{k-1}\omega_{k-1} \\
&= F_{k-1}x_{k-1} + [f(\hat{x}_{k-1}^{+}, u_{k-1}, 0) - F_{k-1}\hat{x}_{k-1}^{+}] + L_{k-1}\omega_{k-1} \\
&= F_{k-1}x_{k-1} + \tilde{u}_{k-1} + \tilde{\omega}_{k-1}
\end{aligned} \tag{2.15}$$

where the superscript + denotes the *a posteriori* estimates. The matrices F_{k-1} and L_{k-1} and the vectors $\tilde{u}_k$ and $\tilde{\omega}_k$ are given by:

$$\begin{aligned}
F_{k-1} &= \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1}^{+}} \\
L_{k-1} &= \left.\frac{\partial f}{\partial \omega}\right|_{\hat{x}_{k-1}^{+}} \\
\tilde{u}_k &= f(\hat{x}_k^{+}, u_k, 0) - F_k\hat{x}_k^{+} \\
\tilde{\omega}_k &\sim (0, L_k Q_k L_k^T)
\end{aligned} \tag{2.16}$$

The same procedure is applied to the measurement equation h:

$$y_k = H_k x_k + z_k + \tilde{\nu}_k \tag{2.17}$$

The matrices and vectors are analogously defined:

$$
\begin{aligned}
H_k &= \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_k^-} \\
M_k &= \left.\frac{\partial h}{\partial \nu}\right|_{\hat{x}_k^-} \\
z_k &= h(\hat{x}_k^-, 0) - H_k \hat{x}_k^- \\
\tilde{\nu}_k &\sim (0, M_k R_k M_k^T)
\end{aligned}
\tag{2.18}
$$

The $-$ superscript denotes the *a priori* estimates. The final model is a linear state space model and the classical Kalman Filter (KALMAN, 1960) can be applied. The Extended Kalman algorithm (EKF) is showed below (SIMON, 2006):

- Filter initialization (at time-step $k = 0$)

$$
\begin{aligned}
\hat{x}_0^+ &= E(x_0) \\
P_0^+ &= E[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T]
\end{aligned}
\tag{2.19}
$$

- At time-steps $k = 1, 2, ...n$, the following algorithm is used for state update:
  - Partial derivatives matrices computation F_{k-1} e L_{k-1}
  - Covariance matrix and state update (*a priori* estimation)

$$
\begin{aligned}
P_{(k|k-1)} &= F_{k-1} P_{(k-1|k-1)} F_{k-1}^T + L_{k-1} Q_{k-1} L_{k-1}^T \\
\hat{x}_{(k|k-1)} &= f(\hat{x}_{(k-1|k-1)}, u_{(k-1)}, 0)
\end{aligned}
\tag{2.20}
$$

  - H_k and M_k computation
  - Convariance matrix and state update (*a posteriori* estimation):

$$
\begin{aligned}
K_k &= P_{(k|k-1)} H_k^T (H_k P_{(k|k-1)} H_k^T + M_k R_k M_k^T)^{-1} \\
\hat{x}_{(k|k)} &= \hat{x}_{(k|k-1)} + K_k[y_k - h(\hat{x}_{(k|k-1)}, 0)] \\
P_{(k|k)} &= (I - K_k H_k) P_{(k|k-1)}
\end{aligned}
\tag{2.21}
$$

2.2.2 Moving Horizon Estimation (MHE) and the Constrained Kalman Filter (CEKF)

The state estimation using the Kalman filter is a fast and low computational cost strategy. It can be noted, through the filter analysis, that the estimated state uses only the data at that time-step, it is known as a null horizon estimator. However, some strategies aim to use the previous data to estimate the states. For example, when a new state is obtained, it is added to a set and the estimation is carried out by using a vector of previous estimations. As one can see the size of this problem increases rapidly, making this strategy, known as batch estimation, inefficient. An alternative proposal, which considers only a fixed amount of data, is the moving horizon estimation, which slides a window of fixed data size along a horizon. When a new measurement is obtained, it is added to the set and the oldest measurement is removed (RAO, 2000). In SALAU (2009), it is presented the MHE algorithm showed below.

Consider the nonlinear discrete-time model (Equation 2.14):

The MHE optimization problem is given by:

$$\begin{aligned}\min_{\omega_{(k-N-1|k)}\cdots\omega_{(k-1|k)}v_{(k-N|k)}\cdots v_{(k|k)}} \Psi_k^N = {} & \hat{\omega}^T_{(k-N-1|k)} P^{-1}_{(k-N-1|k-1)} \hat{\omega}_{(k-N-1|k)} \\ & + \sum_{j=k-N}^{k-1} \hat{\omega}^T_{(j|k)} Q^{-1}_{k-1} \hat{\omega}_{(j|k)} \\ & + \sum_{j=k-N}^{k} \hat{v}^T_{(j|k)} R^{-1}_{(k)} \hat{v}_{(j|k)}\end{aligned}$$

Subject to:

$$\begin{aligned}\hat{x}_{(k-N|k)} &= \hat{x}_{(k-N|k-1)} + \hat{\omega}_{(k-N-1|k)} \\ \hat{x}_{(j+1|k)} &= f(\hat{x}_{(j|k)}, u_j) + \hat{\omega}_{(j|k)} \quad for \quad j = k-N, ..., k-1 \\ y_j &= h(\hat{x}_{(j|k)}) + \hat{v}_{(j|k)} \quad for \quad j = k-N, ..., k \\ \hat{x}_{min} &\leq \hat{x}_{(j|k)} \leq \hat{x}_{max} \\ \hat{\omega}_{min} &\leq \hat{\omega}_{(j-1|k)} \leq \hat{\omega}_{max} \quad for \quad j = k-N, ..., k \\ \hat{v}_{min} &\leq \hat{v}_{(j|k)} \leq \hat{v}_{max} \quad for \quad j = k-N, ..., k\end{aligned} \tag{2.22}$$

Once the NLP (Nonlinear Programming) problem is solved (Equation (2.22)), the vectors ω^* and v^* are obtained and the system state can be evaluated as:

$$
\begin{aligned}
&\hat{x}_{(k-N|k)} = \hat{x}_{(k-N|k-1)} + \hat{\omega}^*_{(k-N-1|k)} \\
&\hat{x}_{(j+1|k)} = f(\hat{x}_{(j|k)}, u_j) + \hat{\omega}^*_{(j|k)} \quad for: \quad j = k-N, ..., k-1 \\
&y_j = h(\hat{x}_{(j|k)}) + \hat{v}_{(j|k)} \quad for: \quad j = k-N, ..., k
\end{aligned}
\tag{2.23}
$$

If we consider a null horizon ($N = 0$) and a linear (or linearized) measurement equation h, the MHE can be solved as a Constrained Kalman Filter using Quadratic Programming (QP), the computational cost is strongly reduced using this formulation. The CEKF equations can be summarized as follows:

$$
\min_{\Theta_{(k|k)}} \hat{\Theta}^T_{(k|k)} S^{-1}_{(k|k)} \hat{\Theta}_{(k|k)} \tag{2.24}
$$

where:

$$
\begin{aligned}
\hat{\Theta}_{(k|k)} &= \begin{bmatrix} \omega_{(k-1|k)} \\ v_{(k|k)} \end{bmatrix} \\
S_{(k|k)} &= \begin{bmatrix} P_{(k-1|k-1)} & 0 \\ 0 & R_k \end{bmatrix}
\end{aligned}
\tag{2.25}
$$

The equality constraints in Equation (2.22) are rewritten as:

$$
[H_k \quad I]\Theta_{(k|k)} = y_k - h(\hat{x}_{(k|k-1)}) \tag{2.26}
$$

The inequality constraints are summarized in the form $A\Theta \leq b$:

$$
\begin{aligned}
A &= \begin{bmatrix} -I \\ I \end{bmatrix} \\
b &= \begin{bmatrix} \hat{x} - x_{min} \\ y_{max} - y_k \\ x_{max} - \hat{x} \\ y_k - y_{min} \end{bmatrix}
\end{aligned}
\tag{2.27}
$$

Despite the lower accuracy of the CEKF formulation, the computational burden is significantly reduced by avoiding the NLP problem and solving the QP formulation.

2.2.3 The *Unscented* Kalman Filter

Using the idea that it is easier to approximate a probability density function than a nonlinear function, JULIER *et al.* (2000) proposed a state estimation strategy for nonlinear systems called Unscented Kalman Filter (UKF). This method is based on the Monte Carlo methodology.

Assuming a model $y = f(x)$, it is possible to obtain the mean of y and x, even if the function f is unknown. To do so, it would be necessary to have a large number of measurements of these two variables and use the usual statistical functions in order to obtain, for example, the mean and covariance matrix.

In online estimation, it would not be possible to perform a large number of x measurements, to calculate the values of y through the function f, and, then, to determine the covariance matrix of y and its mean (due to the short time available to solve the problem). The technique proposed by JULIER *et al.* (2000) consists in obtain a few realizations of the vector x, these realizations being representative, so that the calculated statistics do not distance themselves from those that use a large number of realizations (AGUIRRE, 2007). It is important to note that the proposed transformation is designed to obtain the zero-order and first-order statistical moments, which is sufficient to characterize a normal distribution.

The *Unscented* transformation reduces the large number of realizations required to characterize the moments of y to a small number of values, called sigma points, which are chosen such that:

$$\begin{aligned} \frac{1}{2n_a}\sum_{i=0}^{2n_a}\chi_i &= \bar{\mathbf{x}} \\ cov(\chi) &= \mathbf{P} \end{aligned} \tag{2.28}$$

The first step of the algorithm is to obtain the sigma points and, then, use f to evaluate $\mathcal{Y}_i$ and, finally, obtain the mean and the covariance matrix:

$$\begin{aligned} \frac{1}{2n_a}\sum_{i=0}^{2n_a}\mathcal{Y}_i &\approx \bar{\mathbf{y}} \\ cov(\mathcal{Y}_i) &\approx \mathbf{P}_{yy} \end{aligned} \tag{2.29}$$

Three algorithms are usually used for the sigma points selection and application of the $\mathbf{Q}$ and $\mathbf{R}$ matrices. In the approach proposed by SIMON (2006), $2n$ points are chosen and $\mathbf{Q}$ e $\mathbf{R}$ are added to the system (additive noise). This algorithm is summarized by the following set of equations:

- The algorithm first-step is to obtain the sigma points and use the nonlinear model for time-step propagation

$$\mathcal{X}_{k-1} = [\hat{\boldsymbol{\chi}}_{k-1} + \Gamma\sqrt{\mathbf{P}_{\boldsymbol{\chi}_{k-1}}} \quad \hat{\boldsymbol{\chi}}_{k-1} - \Gamma\sqrt{\mathbf{P}_{\boldsymbol{\chi}_{k-1}}}] \tag{2.30}$$

$$\mathcal{X}_{k,i}^{\boldsymbol{\chi}-} = f(\mathcal{X}_{k-1,i}^{\boldsymbol{\chi}}, \mathbf{u}_{k-1}) \tag{2.31}$$

- *a priori* state and covariance matrix estimation is obtained

$$\hat{\boldsymbol{\chi}}_k^- = \sum_{i=0}^{2n_a} W_i^{\boldsymbol{\chi}} \mathcal{X}_{k,i}^{\boldsymbol{\chi}-} \tag{2.32}$$

$$\mathbf{P}_{\boldsymbol{\chi}_k}^- = \sum_{i=0}^{2n_a} W_i^c (\mathcal{X}_{k,i}^{\boldsymbol{\chi}-} - \hat{\boldsymbol{\chi}}_k^-)(\mathcal{X}_{k,i}^{\boldsymbol{\chi}-} - \hat{\boldsymbol{\chi}}_k^-)^T + \mathbf{Q}_k \tag{2.33}$$

W_i^{χ} and W_i^c are weights that can be adjusted according to the distribution. They are calculated as follows:

$$\begin{aligned}
\lambda &= \alpha^2(n_a + \kappa) - n_a \\
W_0^{\chi} &= \frac{\lambda}{(n_a + \lambda)} \\
W_0^c &= \frac{\lambda}{(n_a + \lambda)} + (1 - \alpha^2 + \beta) \\
W_i^{\chi} = W_i^c &= \frac{1}{2(n_a + \lambda)}
\end{aligned} \tag{2.34}$$

If the system is gaussian, MERWE (2004) proposed:

$$[\alpha \quad \beta \quad \kappa] = [1 \quad 2 \quad 0] \tag{2.35}$$

- Then, the measurement equation is updated by sigma points propagation:

$$\boldsymbol{\gamma}_{k,i} = h(\mathcal{X}_{k,i}^{x-}) \tag{2.36}$$

- It is obtained the predicted measurements at time-step k, the predicted state covariance matrix ($\mathbf{P}_{\mathbf{y}_k\mathbf{y}_k}$) and the cross covariance matrix ($\mathbf{P}_{\mathbf{x}_k\mathbf{y}_k}$):

$$\hat{\mathbf{y}}_k = \sum_{i=0}^{2n_a} W_i^{\boldsymbol{\chi}} \gamma_{k,i} \tag{2.37}$$

$$\mathbf{P}_{\mathbf{y}_k\mathbf{y}_k} = \sum_{i=0}^{2n_a} W_i^c (\gamma_{k,i} - \hat{\mathbf{y}}_k)(\boldsymbol{\gamma}_{k,i} - \hat{\mathbf{y}}_k)^T + \mathbf{R}_k \tag{2.38}$$

$$\mathbf{P}_{\mathbf{x}_k\mathbf{y}_k} = \sum_{i=0}^{2n_a} W_i^c (\mathcal{X}_{k,i}^{\boldsymbol{\chi}-} - \hat{\boldsymbol{\chi}}_k^-)(\boldsymbol{\gamma}_{k,i} - \hat{\mathbf{y}}_k)^T \tag{2.39}$$

- Finally, it is obtained the *unscented* Kalman gain ($\mathbf{K}_k$) and the *a posteriori* state and covariance matrix:

$$\begin{aligned}
\mathbf{K}_k &= \mathbf{P}_{\boldsymbol{\chi}_k\mathbf{y}_k}\mathbf{P}_{\mathbf{y}_k\mathbf{y}_k}^{-1} \\
\hat{\boldsymbol{\chi}}_k &= \hat{\boldsymbol{\chi}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k) \\
\mathbf{P}_{\boldsymbol{\chi}_k} &= \mathbf{P}_{\boldsymbol{\chi}_k}^- - \mathbf{K}_k\mathbf{P}_{\mathbf{y}_k\mathbf{y}_k}\mathbf{K}_k^T
\end{aligned} \tag{2.40}$$

As one can notice, the matrices $\mathbf{Q}$ e $\mathbf{R}$ are added to the system.

In the algorithm proposed by JULIER *et al.* (2000) and MERWE (2004), the state estimated at the previous time-step $\hat{\boldsymbol{\chi}}_{k-1}$ is added to the sigma points set, thus Equation (2.30) is modified such that:

$$\mathcal{X}_{k-1} = [\hat{\boldsymbol{\chi}}_{k-1} \quad \hat{\boldsymbol{\chi}}_{k-1} + \Gamma\sqrt{\mathbf{P}_{\boldsymbol{\chi}_{k-1}}} \quad \hat{\boldsymbol{\chi}}_{k-1} - \Gamma\sqrt{\mathbf{P}_{\boldsymbol{\chi}_{k-1}}}] \tag{2.41}$$

This approach will lead to $2n_a + 1$ sigma points.

MERWE (2004) proposed the augmented state UKF. In this algorithm, the matrices $\mathbf{Q}$ and $\mathbf{R}$ are not added to the system. However, a matrix, which contains $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, is used for the sigma points computation. This augmented matrix is showed in Equation (2.42).

$$\mathbf{P}^a = \begin{bmatrix} \mathbf{P}_{\boldsymbol{\chi}_{k-1}} & 0 & 0 \\ 0 & \mathbf{Q}_{k-1} & 0 \\ 0 & 0 & \mathbf{R}_{k-1} \end{bmatrix} \tag{2.42}$$

Note that this method increases the number of sigma points depending on the number of states and measurements. For example, consider a three state and three measurements, in this case the dimension of $\mathbf{P}^a$ is going to be 9×9, the sum and subtraction of the augmented state $\hat{\boldsymbol{\chi}}_{k-1}^a$ with $\Gamma\sqrt{\mathbf{P^a}_{\boldsymbol{\chi}_{k-1}}}$ will lead to 18 sigma points.

Additionally, if $\hat{\boldsymbol{\chi}}_{k-1}^{a}$ is included, the set will contain 19 sigma points. In the previous algorithm (Equation (2.30)) the set will have 6 points. The augmented state vector is given by $\hat{\boldsymbol{\chi}}_{k-1}^{a} = [\hat{\boldsymbol{\chi}}_{k-1} \;\; 0 \;\; 0]^T$.

2.2.4 Constrained *Unscented* Kalman Filter (CUKF)

The use of constraints in state estimation problems may be a determinant factor for the strategy success. Since these estimators generally depend on dynamic models simulation, unrealistic estimates can lead to problems in model simulation. Another problem, reported in KOLÅS *et al.* (2009), is the existence of multimodal probability density functions with no physical sense, which may cause filter divergence. In the work of VACHHANI *et al.* (2006), a nonlinear optimization problem was proposed to recalculate the filter weights in cases where $\mathcal{X}_{k-1}$ violate some constraint. A more reasonable alternative, where weights do not need to be recalculated, was proposed by KOLÅS *et al.* (2009). In this algorithm, after obtaining the *a priori* estimates $(\hat{\mathbf{y}}_k)$, the *a posteriori* estimates are obtained by minimizing the objective function:

$$J = (\mathbf{y}_k - h(\mathcal{X}_{k,i}^{x}))^T \mathbf{R}_k^{-1} (\mathbf{y}_k - h(\mathcal{X}_{k,i}^{x})) + (\mathcal{X}_{k,i}^{x} - \mathcal{X}_{k,i}^{x-})^T (\mathbf{P}_{\mathcal{X}_k}^{-})^{-1} (\mathcal{X}_{k,i}^{x} - \mathcal{X}_{k,i}^{x-}) \tag{2.43}$$

This is a nonlinear programming problem (NLP), whose computational cost can be high. One way to overcome this problem, it is by using a linear measurement equation, such that:

$$h(\mathbf{x}_k) = \mathbf{D}\mathbf{x}_k \tag{2.44}$$

By substitution in 2.43, we have:

$$J = (\mathbf{y}_k - \mathbf{D}\mathcal{X}_{k,i}^{x})^T \mathbf{R}_k^{-1} (\mathbf{y}_k - \mathbf{D}\mathcal{X}_{k,i}^{x}) + (\mathcal{X}_{k,i}^{x} - \mathcal{X}_{k,i}^{x-})^T (\mathbf{P}_{\mathcal{X}_k}^{-})^{-1} (\mathcal{X}_{k,i}^{x} - \mathcal{X}_{k,i}^{x-}) \tag{2.45}$$

Assuming $\mathbf{R}_k$ e $\mathbf{P}_k$ positive definite and symmetric:

$$\begin{aligned} J = {} & \mathbf{y}_k^T \mathbf{R}_k^{-1} \mathbf{y}_k - 2\mathbf{y}_k^T \mathbf{R}_k^{-1} \mathbf{D}\mathcal{X}_{k,i}^{x} + \mathcal{X}_{k,i}^{Tx} \mathbf{D}^T \mathbf{R}_k^{-1} \mathbf{D}\mathcal{X}_{k,i}^{x} + \mathcal{X}_{k,i}^{Tx} (\mathbf{P}_{\mathcal{X}_k}^{-})^{-1} \mathcal{X}_{k,i}^{x} \\ & - 2\mathcal{X}_{k,i}^{Tx-} (\mathbf{P}_{\mathcal{X}_k}^{-})^{-1} \mathcal{X}_{k,i}^{x} + \mathcal{X}_{k,i}^{Tx-} (\mathbf{P}_{\mathcal{X}_k}^{-})^{-1} \mathcal{X}_{k,i}^{x-} \end{aligned} \tag{2.46}$$

The final problem has the structure:

$$\min_{\mathcal{X}_{k,i}^{x}} J \tag{2.47}$$

However, the previous objective function 2.46 is equivalent to the problem:

$$J^* = -2\mathbf{y}_k^T\mathbf{R}_k^{-1}\mathbf{D}\mathcal{X}_{k,i}^x + \mathcal{X}_{k,i}^{Tx}\mathbf{D}^T\mathbf{R}_k^{-1}\mathbf{D}\mathcal{X}_{k,i}^x + \mathcal{X}_{k,i}^{Tx}(\mathbf{P}_{\mathcal{X}_k}^-)^{-1}\mathcal{X}_{k,i}^x - 2\mathcal{X}_{k,i}^{Tx-}(\mathbf{P}_{\mathcal{X}_k}^-)^{-1}\mathcal{X}_{k,i}^x \tag{2.48}$$

This problem can be solved by using quadratic programming:

$$J^* = \mathcal{X}_{k,i}^{Tx}(\mathbf{D}^T\mathbf{R}_k^{-1}\mathbf{D} + (\mathbf{P}_{\mathcal{X}_k}^-)^{-1})\mathcal{X}_{k,i}^x - 2(\mathbf{y}_k^T\mathbf{R}_k^{-1}\mathbf{D} + \mathcal{X}_{k,i}^{Tx-}(\mathbf{P}_{\mathcal{X}_k}^-)^{-1})\mathcal{X}_{k,i}^x \tag{2.49}$$

Besides the low computational cost, the quadratic programming allows the use of linear constraints:

$$\mathcal{X}_{min} \leq \mathcal{X}_{k,i}^x \leq \mathcal{X}_{max} \tag{2.50}$$

2.3 Stability and robusteness of nonlinear predictive controllers

The stability of predictive controllers is one of the fundamental problems and one of the most studied by the process control community. The issues for the stability proof arise due to the impossibility of applying infinite control and prediction horizons, which leads to a mismatch between the predicted open loop behavior and the resulting closed loop, even in the nominal case.

The guaranteed stability implies a stable controller, regardless of the tuning parameters used. This type of stability, called nominal in absence of plant-model mismatch, is lost when there are uncertainties in the model, disturbances, or imperfect states measurements. Therefore, it is a theoretical formulation that will be used in the robust formulation, in which modeling uncertainties are taken into account, as well as the other factors mentioned above.

Thorough literature reviews about this topic can be found in MAGNI and SCATTOLINI (2007) and MAYNE *et al.* (2000). In this section, we tried to show the main results for the nonlinear MPC problem.

In the work of MAYNE *et al.* (2000), in a survey paper, the literature until 2000s were unified using a uniform notation. The following definitions are using the similar notation proposed later by the same authors in the book RAWLINGS and MAYNE (2013).

A generic formulation for the predictive control problem is given by the minimization of the following optimization problem:

$$V(\mathbf{x}, k, \mathbf{u}) = \sum_{i=k}^{k+N-1} \ell(\mathbf{x}(i), \mathbf{u}(i)) + F(\mathbf{x}(k+N)) \tag{2.51}$$

subject to:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \tag{2.52}$$
$$\mathbf{y}(k) = \mathbf{h}(\mathbf{x}(k)) \tag{2.53}$$
$$\mathbf{u}(k) \in \mathbb{U} \tag{2.54}$$
$$\mathbf{x}(k) \in \mathbb{X} \tag{2.55}$$
$$\mathbf{x}(k+N) \in X_f \subset \mathbb{X} \tag{2.56}$$

The moving horizon strategy apply the first control action from the optimal sequence that includes $N-1$ actions. Thus, given an optimal sequence in $k=0$ denoted by $\mathbf{u}^0(x)$:

$$\mathbf{u}^0(x) = \{\mathbf{u}^0(0;\mathbf{x}), \mathbf{u}^0(1;\mathbf{x}), ..., \mathbf{u}^0(N-1;\mathbf{x})\} \tag{2.57}$$

The implicit control law $\boldsymbol{\kappa_N}$ is given by:

$$\boldsymbol{\kappa_N(x)} = \mathbf{u}^0(0;\mathbf{x}) \tag{2.58}$$

In order to achieve stability, the choice of terminal cost $F(\mathbf{x}(k+N))$ and the terminal set $\mathbf{x}(k+N) \in X_f \subset \mathbb{X}$ are fundamental. The proof that terminal constraints guarantee stability was shown by KEERTHI and GILBERT (1988). Subsequently, RAWLINGS and MUSKE (1993) showed an approximation for the infinite horizon problem that leads to stability for the linear problem. Generally, the stability for the nonlinear controller consists in obtain a Lyapunov function $\mathbb{V}$, an invariant set X, and two class $\mathcal{K}_\infty$[2] functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ and a positive definite function $\alpha_3(\cdot)$, such that $\forall x \in X$ (RAWLINGS and MAYNE, 2013):

[2]A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called class $\mathcal{K}$ if it is continuous, it is zero for zero input, and strictly increasing. Furthermore, if it is unbounded: $\sigma(t) \to \infty \;\; if \;\; t \to \infty$, then it is called class $\mathcal{K}_\infty$.

$$\mathbb{V}(\mathbf{x}) \geq \alpha_1(|\mathbf{x}|) \tag{2.59}$$

$$\mathbb{V}(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|) \tag{2.60}$$

$$\mathbb{V}(\mathbf{x}(k+1)) \leq \mathbb{V}(\mathbf{x}) - \alpha_3(|\mathbf{x}|) \tag{2.61}$$

Consider also the following assumptions:

Assumption 1 (System and cost continuity) *The functions* $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, $l : \mathbb{X} \times \mathbb{U} \to \mathbb{R}+$ *and* $F : X_f \to \mathbb{R}+$ *are continuous, moreover* $f(0,0) = 0$, $\ell(0,0) = 0$ *e* $F(0) = 0$.

Assumption 2 (Set properties) *The sets* $\mathbb{X}$ *and* X_f *are closed,* $X_f \subseteq \mathbb{X}$ *and* $\mathbb{U}$ *is compact. The sets include the origin.*

Assumption 3 (Basic stability) *For every* $\mathbf{x} \in X_f$*, there is a* $\mathbf{u} \in \mathbb{U}$ *such that* $f(\mathbf{x}, \mathbf{u}) \in X_f$.

If these assumptions are satisfied, it is possible to show the following lemma (proof in RAWLINGS and MAYNE (2013), pag. 115):

Lemma 1 (Optimal cost decrease)

$$\forall x \in X_N : \quad V_N^0(f(\mathbf{x}, \kappa_N(x)) \leq V_N^0(\mathbf{x}) - \ell(\mathbf{x}, \kappa_N(\mathbf{x})) \tag{2.62}$$

Thus, the propertie 2.61 is satisfied. In order to define the set X_N, we have to define some auxiliary sets. The set $\mathbb{Z}$ contains the pairs of (x, u) that are inside the problem constraints, $(x, u) \in \mathbb{Z}$. The subscript N denotes a sequence of inputs, therefore $\mathbb{Z}_N(x, \mathbf{u})$ is the set where the constraints are satisfied for the sequence of inputs. Additionally, the set $\mathcal{U}_N$ is composed by the control actions sequences that satisfies state and input constraints:

$$\mathcal{U}_N(x) := \{\mathbf{u} | (x, \mathbf{u}) \in \mathbb{Z}_N\} \tag{2.63}$$

Finally, the set X_N is composed by the states into the set $\mathbb{X}$ and, moreover, the optimization problem has a solution:

$$X_N := \{\mathbf{x} \in \mathbb{X} \ \mid \ \mathcal{U}_N(x) \neq \varnothing\} \tag{2.64}$$

The optimal solution (V_N^0) is given by:

$$V_N^0(\mathbf{x}) = \min_{\mathbf{u}}\{V_N(\mathbf{x},\mathbf{u})|\mathbf{u} \in \mathcal{U}_N(\mathbf{x})\} \tag{2.65}$$

Some remarks about the previous results are necessary. First, the translation to the origin for the equation $x(k+1) = f(x(k), u(k))$ can always be done, as showed by (MACIEJOWSKY, 2000):

$$z(k) = x(k) - x_0 \tag{2.66}$$
$$v(k) = u(k) - u_0 \tag{2.67}$$
$$z(k+1) = x(k+1) - x_0 = f(z(k) + x_0, u(k) + u_0) - f(x_0, u_0) \tag{2.68}$$
$$= g(z(k), v(k)) \tag{2.69}$$
$$0 = g(0,0) \tag{2.70}$$

Second, for each $\mathbf{x} \in X_f$, there is a $\mathbf{u} \in \mathbb{U}$ such that $f(\mathbf{x},\mathbf{u}) \in X_f$, which implies that X_f is control invariant for $\mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)$.

Consider also that the norm used in the cost and terminal cost satisfies the conditions:

Assumption 4 (Bounded cost and terminal cost)

$$\ell(\mathbf{x},\mathbf{u}) \geq \alpha_1(|\mathbf{x}|), \quad \forall \mathbf{x} \in X_N, \forall \mathbf{u} \in \mathbb{U} \tag{2.71}$$
$$F(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|), \quad \forall \mathbf{x} \in X_f \tag{2.72}$$

$\alpha_1(|\mathbf{x}|)$ e $\alpha_2(|\mathbf{x}|)$ belongs to class $\mathcal{K}_\infty$.

By those assumptions, the theorem for MPC nominal stability is shown below:

Theorem 1 (Nominal stability of the MPC) *Suppose, the assumptions (1,2,3,4) hold. Then, the origin is asymptotically stable with region of attraction* X_N *for the system* $\mathbf{x}(k+1) = f(\mathbf{x}(k), \kappa_N(\mathbf{x}))$. *If, additionally, the cost (*ℓ*) and the terminal cost (*F*) satisfies to* $\ell(\mathbf{x},\mathbf{u}) \geq c_1|x|^a$ *for* $\forall \mathbf{x} \in X_n, \quad \forall u \in \mathbb{U}$ *and* $F(\mathbf{x}) \leq c_2|x|^a$, $\forall \mathbf{x} \in X_f$ *for some* $c_1 > 0, \quad c_2 > 0, \quad a > 0$ *and* X_N *is bounded, then the origin is exponentially stable with region of attraction* X_N *for the system* $\mathbf{x}(k+1) = f(\mathbf{x}(k), \kappa_N(\mathbf{x}))$*; If* X_N *is unbounded, then the origin is exponentially stable with region of attraction given by any sublevel of* V_N^0 $\{\mathbf{x}| \ \ V(\mathbf{x}) \leq a \ \}$.

Despite the general results for the NMPC stability, for an effective application, it is necessary to find κ_f (stabilizing control law), X_f (terminal set), and F (terminal cost), that results in a stable optimization problem. Many works in the literature discusses ways for choosing those ingredients.

The simplest way to obtain a stable MPC is the approach proposed by KEERTHI and GILBERT (1988):

$$X_f = 0 \tag{2.73}$$

$$F(0) = 0 \tag{2.74}$$

This choice allows to show, assuming that at the instant $k = 0$ the optimization problem is feasible, that the cost function is Lyapunov and the origin is asymptotically stable, with region of attraction X_N. The use of equality constraint, as pointed out by some authors (ALLGÖWER *et al.*, 2004), has many disadvantages. First, this constraint requires the states to reach the origin in finite time, making the problem impracticable if short prediction horizons are used (the region of attraction becomes very small). Finally, the computational application of terminal constraints can increase the optimization problem burden and may not be solved during the available time .

Another option, which can be used for nonlinear models, is to use a linear local controller around the origin. Therefore, the terminal cost and the terminal set are given by:

$$F(x) = \frac{1}{2}|x|_P^2 \tag{2.75}$$

$$X_f = \{x|F(x) \leq a\} \tag{2.76}$$

In other words, this strategy requires the state trajectory to enter into a origin neighborhood then the control law is modified for a local controller, which can drive the system to the origin. The terminal cost must satisfy the following:

$$F(f(\mathbf{x}(k), \mathbf{u}(k))) + \ell(\mathbf{x}(k), \mathbf{u}(k)) \leq F(\mathbf{x}(k)) \tag{2.77}$$

The Equation 2.77 implies that terminal cost is a control Lyapunov function (CLF).

For linear controllers, it is possible to use an infinite horizon approximation for the terminal cost, leading to stability, as shown by RAWLINGS and MUSKE (1993). The key idea is to parameterize the infinite horizon problem in finite terms. The infinite horizon problem, theoretically, has the following objective function (for the regulation problem and Euclidean norm)[3]:

$$V(k) = \sum_{j=1}^{\infty} ||\mathbf{x}(k+j|k)||^2_{\mathbf{Q}} + ||\mathbf{u}(k+j|k)||^2_{\mathbf{R}} \tag{2.78}$$

for the linear system:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{2.79}$$

Only the first “m ”control actions are non zero:

$$u(k+j|k) = 0, i > m \tag{2.80}$$

By using the constraint (Equation 2.80), the objective function is rewritten as:

$$V(k) = \underbrace{\sum_{j=0}^{\infty} ||\mathbf{x}(k+j|k)||^2_{\mathbf{Q}}}_{T} + \sum_{j=0}^{m-1} ||\mathbf{u}(k+j|k)||^2_{\mathbf{R}} \tag{2.81}$$

The first infinite term must be rewritten:

$$T = \sum_{j=0}^{m} ||\mathbf{x}(k+j|k)||^2_{\mathbf{Q}} + \underbrace{\sum_{j=m+1}^{\infty} ||\mathbf{x}(k+j|k)||^2_{\mathbf{Q}}}_{T2} \tag{2.82}$$

T2 can be written as:

$$T2 = \sum_{j=0}^{\infty} ||\mathbf{x}(k+j+m|k)||^2_{\mathbf{Q}} = \sum_{j=0}^{\infty} \mathbf{x}(k+m+j|k)^T \mathbf{Q}\mathbf{x}(k+m+j|k) \tag{2.83}$$

By using the model for prediction:

$$T2 = \mathbf{x}(k+m|k)^T \underbrace{\left[\sum_{j=0}^{\infty} (\mathbf{A^T})^{\mathbf{j}} \mathbf{Q} \mathbf{A^j}\right]}_{\overline{\mathbf{Q}}} \mathbf{x}(k+m|k) \tag{2.84}$$

The term $\overline{\mathbf{Q}}$ can be recalculated as:

[3]The notation was slightly modified to comply with the infinite horizon literature. $\mathbf{x}(k+j|k)$ is the prediction of $\mathbf{x}$ up to $k+j$ at time-step k.

$$\mathbf{A^T \bar{Q} A} = \sum_{j=0}^{\infty} (\mathbf{A^T})^{\mathbf{j}} \mathbf{Q} \mathbf{A^j} = \mathbf{\bar{Q}} - \mathbf{Q} \tag{2.85}$$

This equation (known as Lyapunov Equation) can be solved for $\overline{\mathbf{Q}}$, given $\mathbf{Q}$ and $\mathbf{A}$. Using $\overline{\mathbf{Q}}$ the cost is given by:

$$V(k) = \sum_{j=0}^{m} ||\mathbf{x}(k+j|k)||^2_{\mathbf{Q}} + \sum_{j=0}^{m-1} ||\mathbf{u}(k+j|k)||^2_{\mathbf{R}} + \underbrace{||\mathbf{x}(k+m|k)||^2_{\overline{\mathbf{Q}}}}_{terminal \quad cost} \tag{2.86}$$

The nominal stability is assured for any m, $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} \geq \mathbf{0}$. Many authors have used the infinite horizon approach for predictive controllers formulation, for example we have the works of PEREZ *et al.* (2014), MARTINS *et al.* (2013),ODLOAK (2004) and, more recently, for unstable systems, MARTINS and ODLOAK (2016).

Robust Predictive Control

Despite the effort for a guaranteed stability controller, in a real plant, which contains model uncertainties, disturbances, non-measured states and corrupted measurements, this theory must be complemented to cope with a large range of situations and preserve stability.

The first step for a robust controller design is the uncertainty description for the controller and state estimation. In general, the system under uncertainty is described by using a set of models or, as previously, using a variable for the uncertainty vector ($\mathbf{w}$):

$$\mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k), \mathbf{w}(k)) \tag{2.87}$$

The uncertainty vector is always bounded, and belongs to the set $\mathbb{W}$ (RAWLINGS and MAYNE, 2013). The uncertainty description is fundamental, since an incorrect assumption may result in instability such as in the nominal case. Despite the boundness in $\mathbf{w}$, usually, it is assumed unknown. Thus, given the control action and the system state, the predictions can not be described by single trajectory. The trajectories inclusion set is denoted by $\mathbb{F}$, which some authors use the denomination "tubes". As one can see, once the uncertainty is added to the system an operator maps a set of trajectories from one time-step to the next one. The definition for robust global asymptotic stability is given by (TEEL, 2004):

Definition 2.3.1 (Robust Global Asymptotic Stability (GAS)) *Given $\mathbb{A}$ compact, the distance between this set and a point in the state space $\mathbf{x}$ is given by $d(\mathbf{x}, \mathbb{A}) := min_a\{|a - x||a \in \mathbb{A}\}$ and denoted by $|\mathbf{x}|_{\mathbb{A}}$. The set $\mathbb{A}$ is GAS for the system $\mathbf{x}(k+1) = f(\mathbf{x}(k))$ if there is a function β belonging to the class $\mathcal{KL}$[4], such that, for each $\epsilon > 0$ and each set $\mathbb{C}$ compact, there is a $\delta > 0$ such as, for each $\mathbf{x} \in \mathbb{C}$ and every solution of the difference equation $\Phi \in S_\delta(\mathbf{x})$[5], the following statement is valid:*

$$|\Phi(k; \mathbf{x})|_{\mathbb{A}} \leq \beta(|\mathbf{x}|_{\mathbb{A}}, k) + \epsilon \quad \forall k \in \mathbb{I}_+$$

The rigorous solution of a nonlinear robust control problem would require a computational burden that will make the problem unfeasible for a real time solution. Some proposals, by sacrificing optimality, have tried to simplify the problem in order to make the problem feasible for a real time solution. Among the approaches used for nonlinear systems, we can cite the "worst case" or min-max approach, the tubes strategy, the multi-stage, and the Lipschitz approach[6].

For the min-max approach, MORARI and CAMPO (1987) proposed that the system must operate in a way that the worst-case scenario is the most likely to occur. For example, the control actions are such that the constraints are satisfied even if the disturbances are the worst possible. Given the objective function showed in 2.51, the uncertain problem 2.87 is formulated in the following way:

$$\min_{\mathbf{u}} \max_{\mathbf{w} \in \mathbb{W}} V(\mathbf{x}, k, \mathbf{u}, \mathbf{w}) \tag{2.88}$$

ALLGÖWER *et al.* (2004) mention some problems that may occur in this formulation. First, the solution for the optimization problem may not exist[7]. Second, the control action is optimal for the worst-case scenario, which may not have a high occurrence probability. Furthermore, the application of the control action in most likely scenarios may result in poor performance. Finally, the stability is guaranteed when the uncertainties are well defined, which may be difficult in some cases.

An alternative proposal is the tube approach. MAYNE *et al.* (2011) propose a strategy for linear systems by using two MPC controllers, the first one uses

[4]the function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to $\mathcal{KL}$ if it is continuous and if $\beta(s, t)$ is class $\mathcal{K}$ in respect to time and decreasing in respect to s and, furthermore, goes to zero if $t \to \infty$

[5]$S_\delta(\mathbf{x})$ is the set of the disturbed system solutions $\mathbf{x}(k+1) = f(\mathbf{x}(k) + \mathbf{e}(k)) + \mathbf{w}(k)$ such that $max\{||\mathbf{x}||, ||\mathbf{u}||\} \leq \delta$

[6]This method is described and used in Chapter 4

[7]The existence of an optimal solution is given by the Weierstrass' Theorem, which requires compactness of both optimization sets and continuity of the objective function, see (BERTSEKAS *et al.*, 2003)

tight constraints for the nominal trajectory, the second, called ancillary controller, drives the uncertain system for the nominal trajectory. MAYNE *et al.* (2011) have considered the following additive disturbance:

$$\mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)) + \mathbf{w}(k) \tag{2.89}$$

subject to:

$$\mathbf{w} \in \mathbb{W} \tag{2.90}$$

$$\mathbf{x} \in \mathbb{X} \tag{2.91}$$

$$\mathbf{u} \in \mathbb{U} \tag{2.92}$$

The nominal system is given by:

$$\mathbf{z}(k+1) = f(\mathbf{z}(k), \mathbf{v}(k)) \tag{2.93}$$

The deviation between the actual trajectory and the nominal system is:

$$\mathbf{e}(k+1) = \mathbf{x}(k+1) - \mathbf{z}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)) - f(\mathbf{z}(k), \mathbf{v}(k)) + \mathbf{w}(k) \tag{2.94}$$

The nominal controller is the same as the one in previous section, the nominal system under its control satisfies the equation:

$$\mathbf{z}(k+1) = f(\mathbf{z}(k), \boldsymbol{\kappa}(\boldsymbol{z})) \tag{2.95}$$

In this case, the terminal cost and the terminal set ($\mathbb{Z}_f$) are chosen by using the previous stability criteria. Moreover, the ancillary controller objective is to drive the uncertain system to the nominal trajectory, therefore, the following objective function is used:

$$V_N(\mathbf{x}, \mathbf{z}, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(\mathbf{x}(k) - \mathbf{z}^*(k; z), \mathbf{u}(k) - \mathbf{v}^*(k; z)) \tag{2.96}$$

Where $\mathbf{z}^*(k; z)$ e $\mathbf{v}^*(k; z)$ are the optimal trajectory and the optimal control actions from the nominal MPC. The ancillary controller is subject to the terminal constraint:

$$\mathbf{x}(N) = \mathbf{z}^*(N; \mathbf{z}) \tag{2.97}$$

It is possible to prove that this strategy satisfies the robust stability conditions using some assumptions, details are found in MAYNE *et al.* (2011) and RAWLINGS and MAYNE (2013).

In the last years, some new ideas are found in the literature seeking a less conservative robust controller. One prominent approach is the multi-stage method (LUCIA *et al.*, 2013). This controller uses discrete scenarios for the disturbances in every time-step, which can be updated in the future time-steps, improving the controller performance.

2.4 MPC feedback and state estimation

The control of a dynamic system can be viewed as a problem of obtaining a function $\mathbf{u}(\cdot)$ that depends on the state or system outputs. Given a dynamic system:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \boldsymbol{\theta}, \mathbf{u}) \tag{2.98}$$

since, in industrial process systems, the state vector $\mathbf{x}$ is usually not fully measured, a mathematical relationship between the output $\mathbf{y}$ and the system states is used:

$$\mathbf{y} = h(t, \mathbf{x}) \tag{2.99}$$

The most used predictive control formulation is the state feedback for model prediction. In this approach, an observer is used for estimate the system state by using the available measurements. In this case, the control law can be represented by the following function:

$$\mathbf{u} = \kappa(\hat{\mathbf{x}}) \tag{2.100}$$

Usually in MPC, as soon as a new measurement is available, the state estimation problem is solved and used as a initial condition for the dynamic model prediction. Therefore, the problems are coupled. It is possible to show that even if both problems are stable the coupled problem may be unstable. (FINDEISEN *et al.*,

2003) mentioned two mechanisms for solving this problem:

- The first approach is to separate the observer error and the state feedback error. The most used way is by using fast convergence observers, in which the error convergence rate results in a very small absolute error.
- The second strategy employs observers that provide uncertainties estimates that can be explicitly used in a robust formulation.

The last approach is used for robust formulation in Chapter 4 for the adaptive problem and Chapter 5 for the output feedback problem. In Section 5.2, a specific literature review regarding this estimators is presented.

2.5 Adaptive Control

ASTRÖM and WITTENMARK (2008) defined an adaptive controller as any controller that has adjustable parameters and a strategy or algorithm for its tuning.

Many aspects of this thesis are related to adaptive control, in which the control law or the MPC plant model are modified. Adaptive control techniques can be divided in three classes (OGUNNAIKE and RAY, 1994): scheduled control, model reference adaptive control (MRAC) and self-tuning control. In the end of this section, we also present the dual control theory that generalizes these strategies in adaptive control.

2.5.1 Scheduled control

In this approach, the process variables can be related with the plant parameters and, finally, with the controller parameters. In this case, the function that relates the plant variables and the control parameters is obtained *a priori*, therefore, the plant model must be accurate. The most widespread application of this strategy is the gain scheduling, which uses a process variable to indirectly estimate the process gain and adjust the controller gain (ASTRÖM and WITTENMARK, 2008) (OGUNNAIKE and RAY, 1994). Figure 2.1 shows the scheduled control scheme.

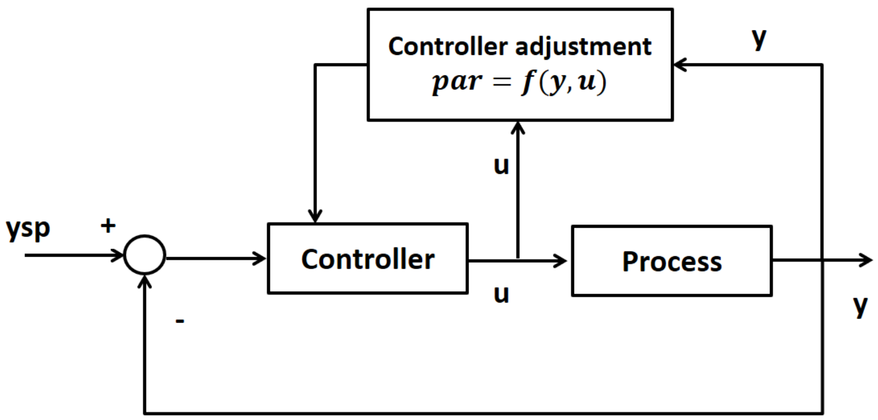

Figure 2.1: Scheduled control scheme.

2.5.2 Model reference adaptive control

In this adaptive strategy, a reference model is used for performance monitoring and predicts how the plant should response ideally. In Figure 2.2, it is showed the MRAC scheme, the internal loop is a feedback loop, the external loop adjust the controller parameters using the error between the reference model and the plant.

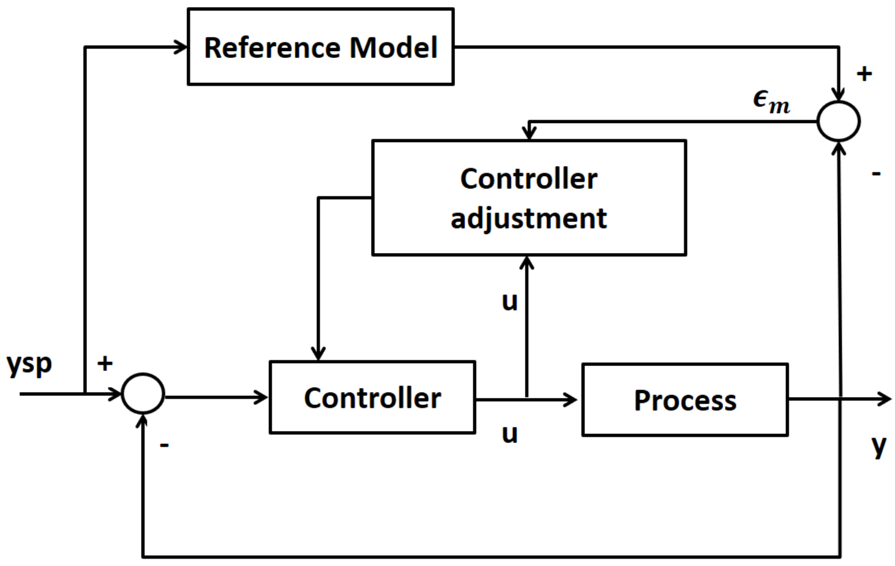

Figure 2.2: Model reference adaptive control scheme.

In the MRAC strategy, the control law is a sensitive issue. For a long time, the rule known as "MIT rule" have been used for adaptation:

$$\frac{d\theta}{dt} = -\gamma e \frac{\partial e}{\partial \theta} \tag{2.101}$$

In Equation 2.101, the error is given by $e = y - y_m$, the difference between the plant output and the model. This rule is a gradient method for minimizing the quadratic error (ASTRÖM and WITTENMARK, 2008). Despite its popularity, PARKS (1966) showed, in a work that would become one of the most important in adaptive control theory, that "MIT rule" could make unstable several systems, including simple systems such as a second order system disturbed by a sinusoidal signal. In addition, Parks proposed another method to determine the adjustment law based on the second Lyapunov method, showing that, for positive and real functions, the choice of a candidate Lyapunov function of the type:

$$V = \mathbf{e}^T \mathbf{P} \mathbf{e} + \boldsymbol{\lambda} \mathbf{x}^2 \tag{2.102}$$

and using the Kalman Lemma (see KHALIL (2002)) it was possible to achieve stability.

2.5.3 Self-tuning adaptive control

In the self-tuning control, the input and output process data are used in a parameter estimation routine and, thereafter, for tuning the controller. In Figure 2.3 is showed the self-tuning scheme.

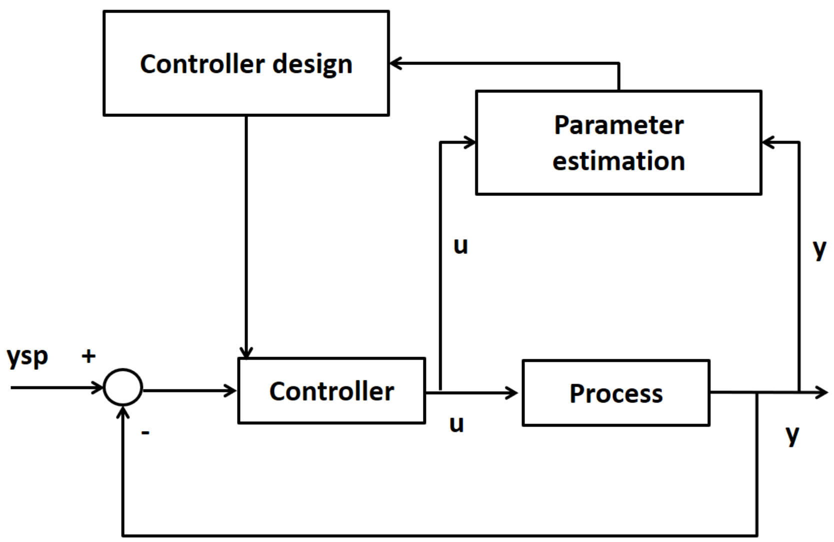

Figure 2.3: Self-tuning control scheme

It is noted that in the two previous methods shown above the process data was obtained a priori. Differently, in the self-tuning method, plant information can be obtained after the controller start-up. Despite the low prior knowledge of the plant needed for adaptive controller design, this type of system needs some details for proper operation, such as a reliable parameter estimation method in which lack of excitation in the system does not result in instability (ASTRÖM and WITTENMARK, 2008).

2.5.4 Dual Control

The aforementioned schemes do not use parametric uncertainties at any stage of the controller design. However, it is possible to notice that the parameter estimates will not always be of the same quality, since they depend, for example, on the measurements accuracy and the system excitation level. In the dual control problem, system parameters are added to the states, creating an augmented vector:

$$\mathbf{z}(t) = [\mathbf{x}(t) \;\; \boldsymbol{\theta}(t)]^T \tag{2.103}$$

The dual strategy aims to obtain, through the estimator, the probability density function of the increased vector, which is called a hyperstate. It can be obtained in simplified cases or through the solution of a nonlinear state estimation optimization problem. Because the hyperestate needs to be updated frequently, the exact real-time solution in general is unfeasible (ASTRÖM and WITTENMARK, 2008). The dual control minimizes the function:

$$V = E\left(G(\mathbf{z}(T), \mathbf{u}(T)) + \int_0^T g(\mathbf{z}, \mathbf{u})dt\right) \tag{2.104}$$

G and g are scalar functions, and E is the expectation operator that must be calculated with respect to the distribution of all initial values and all disturbances that appear in the system model. The V function is minimized in relation to the control variables $\mathbf{u}$.

This problem can be seen as the combination of a non-linear estimator and a feedback controller. The estimator generates the states probability distribution from the available measurements and the controller can be seen as a nonlinear function that maps these estimates onto the manipulated variables space.

This problem solution is a combination between reducing the parameters uncertainty and maintaining or directing the controlled variable to the desired value. The dual problem can be simplified by obtaining only the mean and covariance matrix of the system, for more details about this approach see ASTRÖM and WITTENMARK (2008).

2.6 Offset in model predictive control

One of the fundamental aspects for MPC practical application is the steady-state error elimination, known as *offset*[8]. Many proposals are based on a disturbance model, which must be estimated since unmeasured disturbances are present, in addition to the modeling uncertainties. In constrained control problems, such as MPC, adding an integrator does not necessarily result in offset-free, as shown in PANNOCCHIA and RAWLINGS (2003).

Consider the nonlinear-discrete-time model for the plant:

$$\begin{aligned} \mathbf{x}(k+1) &= f(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) &= h(\mathbf{x}(k)) \end{aligned} \tag{2.105}$$

this model can be augmented for a disturbance incorporation ($\mathbf{d}(k)$):

$$\begin{aligned} \mathbf{x}(k+1) &= f_a(\mathbf{x}(k), \mathbf{d}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) &= h_a(\mathbf{x}(k), \mathbf{d}(k)) \\ \mathbf{d}(k+1) &= \mathbf{d}(k) \end{aligned} \tag{2.106}$$

This additive disturbance may not represent the true disturbances acting in the plant, its purpose is to represent the non modeled system dynamics, the unknown disturbances, the parametric uncertainties, the structural uncertainties and the modeling errors (MORARI and MAEDER, 2012). However, this structure is also solved as a state and disturbance estimation using an observer:

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= f_a(\hat{\mathbf{x}}(k), \hat{\mathbf{d}}(k), \mathbf{u}(k)) + \ell_x(\mathbf{y}(k) - h_a(\hat{\mathbf{x}}(k), \hat{\mathbf{d}}(k)) \\ \hat{\mathbf{d}}(k+1) &= \hat{\mathbf{d}}(k) + \ell_d(\mathbf{y}(k) - h_a(\hat{\mathbf{x}}(k), \hat{\mathbf{d}}(k)) \end{aligned} \tag{2.107}$$

where ℓ_x is an output to state map ($\ell_x : \mathcal{Y} \rightarrow \mathcal{X}$) and ℓ_d an output to disturbance map, in both cases $\ell_x(0) = \ell_d(0) = 0$.

The conditions for an offset free MPC were outlined by MORARI and MAEDER (2012) based on the linear case proposed by PANNOCCHIA and RAWLINGS (2003). The conditions are:

- The length of the disturbances must be of the same dimension of the input

[8] In a few applications, such as level control for oscillation attenuation, an offset free controller may not be suitable

variables and the same of the output variables ($n_y = n_u = n_d$)

- Local observability for the augmented system in the steady-state
- Local controlability for the augmented system in the steady-state
- The observer, in the nominal case, must be designed for steady-state offset-free
- It is possible to design an offset-free controller that is nominally steady-state offset-free
- The plant converges to a single point steady-state in closed-loop

The disturbance model choice is essential to ensure that any output from the real system can be generated through the augmented model, therefore the chosen model should be rich enough to do so. The model must ensure that the nonlinear system is observable, which is not a trivial task (See for an example GONÇALVES *et al.* (2014b)).

The two most commonly used approaches for the disturbance model are the pure output and pure input models. In the pure output model, an artificial state is added for each system output, resulting in the nonlinear model:

$$\begin{aligned}
\mathbf{x}(k+1) &= f(\mathbf{x}(k), \mathbf{u}(k)) \\
\mathbf{d}(k+1) &= \mathbf{d}(k) \\
\mathbf{y}(k) &= h(\mathbf{x}(k)) + \mathbf{d}(k)
\end{aligned} \tag{2.108}$$

This type of model has the advantage of being able to generate any system output through the disturbance model. The industrially used model, known as bias-type correction or DMC-type correction (in reference to the *Dynamic Matrix Controller*), calculates the disturbance using the equation:

$$\mathbf{d}(k) = \mathbf{y}(k) - \mathbf{y}_p(k|k-1) \tag{2.109}$$

where $\mathbf{y}_p$ is the predicted output.

According to MUSKE and BADGWELL (2002), the pure output modeling strategy can result in poor models for the real disturbances into the plant, leading to oscillatory behaviors. In the pure input method, the disturbance is combined with the inputs, being evaluated through the model, resulting in a more realistic model:

$$
\begin{aligned}
&\mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k) + \mathbf{d}(k)) \\
&\mathbf{d}(k+1) = \mathbf{d}(k) \\
&\mathbf{y}(k) = h(\mathbf{x}(k))
\end{aligned} \tag{2.110}
$$

The pure input model generally results in a better performance in comparison with the pure output. In addition, if the plant contains integrating modes, the pure output model has no guaranteed observability (MORARI and MAEDER, 2012).

In the present work, one of the proposed approaches is to augment the state vector with the parameter vector and to use an observer to estimate both (Chapter 3). Therefore, it is a strategy for modifying the function $f(\mathbf{x}(k), \mathbf{u}(k))$. The control law should, in theory, contemplate a parametric set that is capable of generating any steady-state output of the real plant. One difficulty of applying this strategy is that the parametric set can generate several steady-states for the same set of inputs, which can result in offset. Another difficulty is to satisfy the observability criterion for the augmented state, since, at least, it must have a disturbance for each controlled variable, which may result in large observers. The problem can be formulated as follows, given the real plant (subscript r):

$$
\begin{aligned}
&\mathbf{x}_r(k+1) = f_r(\mathbf{x}_r(k), \mathbf{u}_r(k), \boldsymbol{\theta}_r) \\
&\mathbf{y}_r(k) = h_r(\mathbf{x}_r(k))
\end{aligned} \tag{2.111}
$$

a model for this plant is:

$$
\begin{aligned}
&\mathbf{x}_m(k+1) = f_m(\mathbf{x}_m(k), \mathbf{u}_m(k), \boldsymbol{\theta}_m) \\
&\mathbf{y}_m(k) = h_m(\mathbf{x}_m(k))
\end{aligned} \tag{2.112}
$$

Given that the $\boldsymbol{\theta}_r$ vector and the f_r function are unknown, one can ask: is there enough flexibility that for a partial selection of vector elements $\boldsymbol{\theta}_m \in \mathbb{W}$, it is possible to generate all real plant steady-states $(\mathbf{u}_r(\infty), \mathbf{y}_r(\infty))$? Additionally, the model must preserve observability and avoid steady-state multiplicity, so that the conditions exposed in MORARI and MAEDER (2012) are satisfied. In Chapter 3, it is shown, using simulations, that the correct choice of the parametric set can result in an offset-free controller even though it is not known which parameter is uncertain or if the plant is subject to unmeasured disturbances. Moreover, it is showed, for a nonlinear case study, that the pure-input model using a parameter approach is superior to the bias-type pure-output correction. Finally, it is showed that the latter may also result in an unstable system in NMPC.

Chapter 3

Adaptive MPC using Kalman Filters: current algorithms analysis[1]

3.1 Introduction

Despite the large development of Kalman Filters observers, a comprehensive analysis of these methods for joint estimation (parameter and states) and its application in MPC is still lacking. In this chapter, the most used algorithms and its variations are used for state and parameter estimation. Furthermore, the Kalman filter application for adaptive MPC is addressed in a benchmark control problem.

After the seminal work of KALMAN (1960), many algorithms have been developed for improvements, such as nonlinear estimation application and use of constraints. The first, and most used, algorithm for nonlinear systems applications is the Extended Kalman Filter (EKF) (see Section 2.2.1). In this algorithm, a model linearization is carried out every time-step, then the classical algorithm for linear systems can be used. This algorithm performance is dependent of linearization quality, since it can be obtained analytically or numerically. For complex or large systems that depends on thermodynamics relationships, such as distillation columns, the analytical linearization is a difficult task (GONÇALVES *et al.*, 2014b).

An alternative, for direct nonlinear system application and to avoid a nonlinear programming optimization, is the use of Unscented Kalman Filters (JULIER *et al.*, 2000). In this algorithm, the estimation is done by sequential nonlinear system simulations. However, analysis for joint estimation are difficult to find in the literature. Furthermore, we analyze a recent algorithm by KOLÅS *et al.* (2009)

[1]Some simulations in this chapter were presented in COBEQ 2014 GONÇALVES *et al.* (2014a) in a comparison with other NMPC strategies

for constrained UKF estimation and its aplication for joint estimation. Finally, a method for numerically reducing the failures in the UKF algorithm under excitation absence is proposed.

We then proceed to the use of Kalman Filters for adaptive MPC. In the work of KLATT and ENGELL (1998), some adaptive strategies, such as exact linearization and gain-scheduling, were used for a reactor control. The same model was used as a benchmark for control systems in an adaptive MPC using a sequential optimization approach.

The remainder of this chapter is organized as follows: in Section 3.2, the most used estimators are evaluated aiming joint estimation; in Section 3.3, the numerical aspects for solving the MPC problem are detailed; in Sections 3.4, 3.5 and 3.6 the Kalman Filter is used for tests in nonlinear predictive control; in Section 3.7, the final remarks are presented.

3.2 Estimators analysis

Initially, for state estimation, five (most used) algorithms were compared:

- The extended Kalman Filter (EKF)
- The constrained Kalman Filter (CEKF)
- The Unscented Kalman Filter (UKF)
- The Unscented Kalman Filter with constraints (CUKF)
- The augmented state Unscented Kalman filter (CUKF-FA)

These algorithms were described in Section 2.2.

The tests were conducted on a CSTR with the Van Vusse reaction. This system is widely used as benchmark for controller testing due to inverse response, overshoot and nonlinear behaviors such as gain sign inversion (TRIERWEILER and SECCHI, 2000). The reaction system represents the synthesis of cyclopentanol (B) using cyclopentadiene (A) as the reagent. Undesirable by-products are cyclopentenediol (C) and dicyclopentadiene (D). The system is described by a two series reactions and one in parallel as shown in Equation (3.1):

$$\begin{aligned} &A \xrightarrow{k1} B \xrightarrow{k2} C \\ &2A \xrightarrow{k3} D \end{aligned} \tag{3.1}$$

The energy and mass balance are given by:

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{Ae} - C_A) - k_1 C_A - k_3 C_A^2 \tag{3.2}$$

$$\frac{dC_B}{dt} = \frac{-F}{V} C_B + k_1 C_A - k_2 C_B \tag{3.3}$$

$$\frac{dT}{dt} = \frac{1}{\rho C_p}\left[k_1 C_A(-\Delta H_{AB}) + k_2 C_B(-\Delta H_{BC}) + k_3 C_A^2(-\Delta H_{AD})\right] + \frac{F}{V}(T_0 - T) + \frac{K_w A_R}{\rho c_p V}(T_K - T) \tag{3.4}$$

The desired product, as mentioned above, is component B. The control objective is to keep component B at its maximum value by manipulating the reactor inlet flow rate (dilution rate). However, the maximum concentration of B is exactly at one of the points where the gain is reversed. Figure 3.1 [2] shows the steady-state concentration of component B as a function of the dilution rate. The parameters used in this work can be found in KLATT and ENGELL (1998).

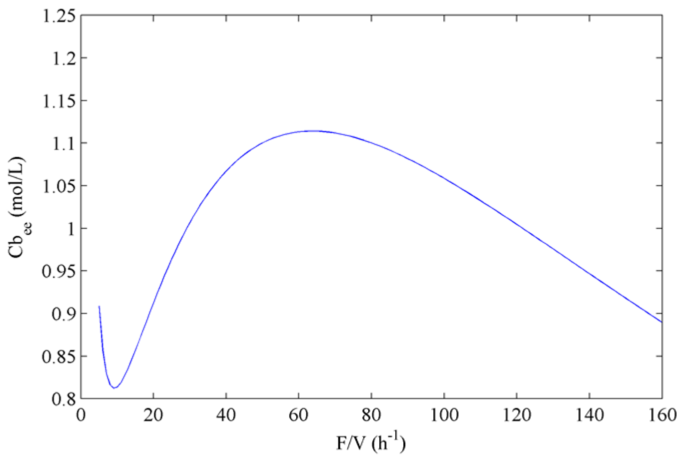

Figure 3.1: C_B as a function of the dilution rate

The first step was a comparison between the estimators, using the constrained and unconstrained versions. On one hand, the unconstrained algorithms have the advantage of dismiss a routine to solve the optimization problem. On the other hand, the constrained estimation algorithms result in a quadratic optimization problem for a linear measurement equation. These algorithms were implemented using the software Matlab (MATLAB, 2008) and its *quadprog* routine. The only available measurement for simulation was the reactor temperature, which was corrupted using a Gaussian noise of 1% of the nominal value. In Figure 3.2 the filtering results for

[2] The first point of gain sign inversion, at low dilution rates, depends on the value used for the cooling jacket temperature. In Section 3.4, the cooling jacket energy balance is inserted for control purposes, see Equation 3.16. Figure 3.1 was generated using the most generic model, including the jacket energy balance

the temperature measurements are shown, revealing that the filters have similar performance.

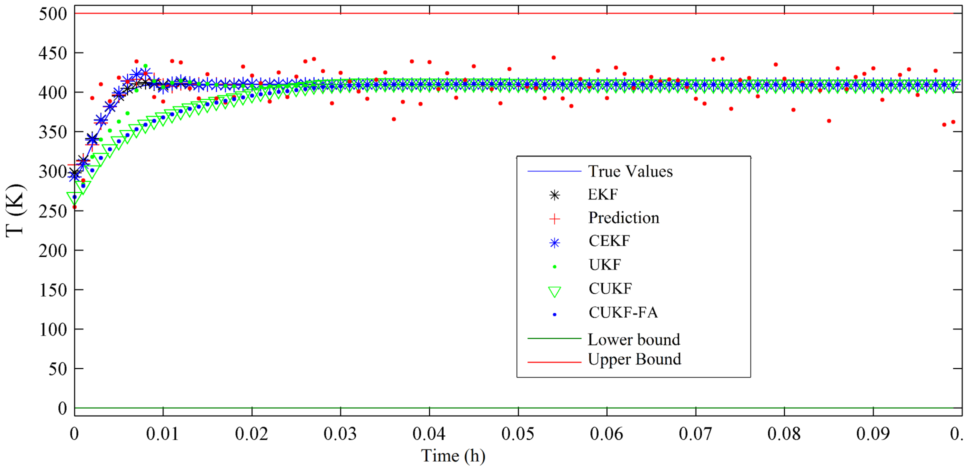

Figure 3.2: Temperature estimation

The results for the unmeasured variables estimation (concentrations of components A and B) are shown in Figures 3.3 and 3.4. These figures reveal that unconstrained filters can cause oscilations even in low noise situations. The UKF filter violated physical constraints for the concentration of component A, leading to results where the estimated composition was higher than the feed composition (Figure 3.3). The constrained filters results were superior in comparison with the same formulations without constraints. The convergence was always faster in the constrained algorithms, in agreement with the results presented by KOLÅS *et al.* (2009). In Figure 3.4, the constrained filters higher performance is quite pronounced during the transient period, while the UKF lead to A and B composition levels above the constraints. For temperature filtering (Figure 3.2), the results were similar among the estimators, also presenting a similar convergence velocity.

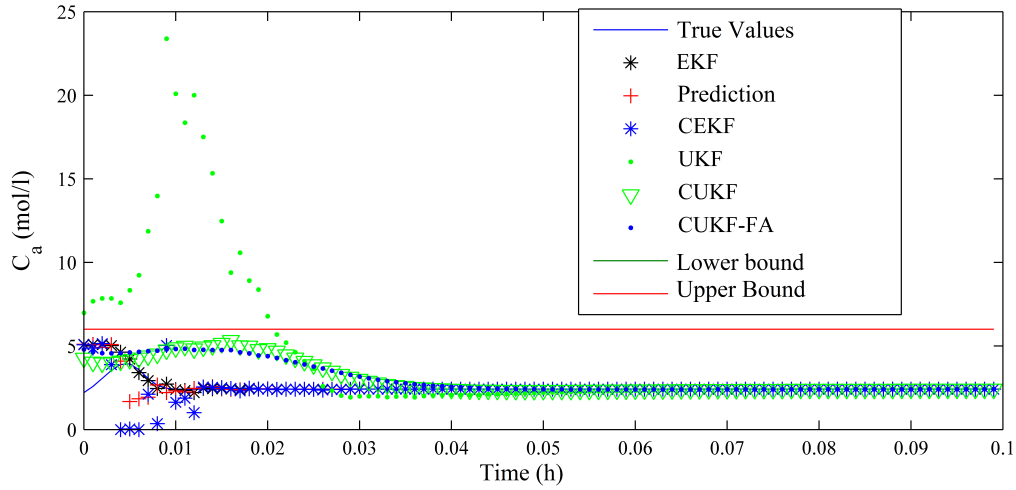

Figure 3.3: Component A concentration estimation

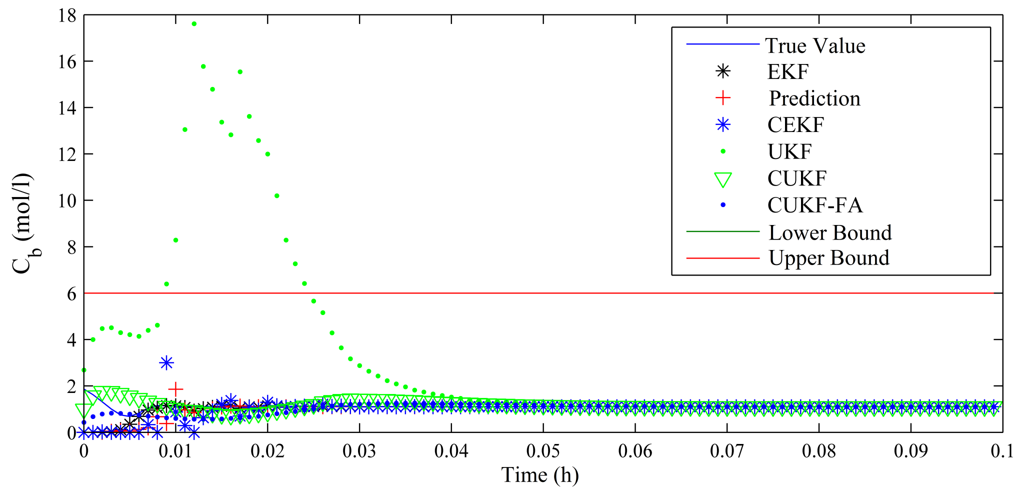

Figure 3.4: Component B concentration estimation

Due to the higher performance of the constrained filters, for the joint estimation (state and parameters) only the constrained versions were used.

The state vector was augmented for the parameter inclusion. In this case, the feed temperature was chosen as the parameter to be estimated. The extended, discrete-time model, under the assumption that the parameter dynamic is slower than the states can be represented by the equation:

$$\mathbf{x}_a = \begin{bmatrix} \hat{\mathbf{x}}_{k+1}^{+} \\ \hat{\theta}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_{k}^{+} \\ \hat{\theta}_{k} \end{bmatrix} \tag{3.5}$$

where:

$$\hat{\mathbf{x}}_{k+1}^{+} = [C_A \quad C_B \quad T]^T$$
$$\hat{\theta}_{k+1} = T_0 \tag{3.6}$$

The problem was subject to the following constraints:

$$\mathbf{x}_a(min) = [0 \quad 0 \quad 0 \quad 250]^T \tag{3.7}$$
$$\mathbf{x}_a(max) = [6 \quad 6 \quad 500 \quad 500]^T \tag{3.8}$$

The initial conditions for the estimator and the plant were distinct and given by:

$$\mathbf{x}_a^{plant}(0) = [2 \quad 2 \quad 275 \quad 383]^T \tag{3.9}$$

$$\mathbf{x}_a^{filter}(0) = [5.1 \quad 0 \quad 298 \quad 390]^T \tag{3.10}$$

The joint estimation results are quite different, the UKF convergence is slower in comparison with the constrained Kalman estimator. Figure 3.5 shows the estimation results for component A composition. As one can see, the UKF and UKF-FA filters do not converge to the true parameter value, leading to an offset, the same occurs for the component B (Figure 3.7). For the temperature, the offset is reduced. Finally, the parameter estimate (feed temperature) has its time evolution presented in Figure 3.8. In this case, it is noticed that the UKF filters have performed worse than the CEKF filter, presenting a high overshoot and offset, while the CEKF filter showed an initial oscillation with subsequent convergence to the true value.

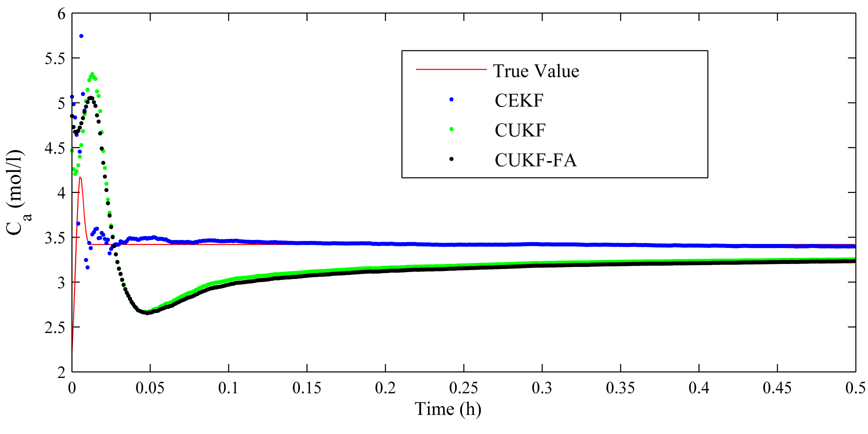

Figure 3.5: C_A composition under joint estimation using the constrained filters.

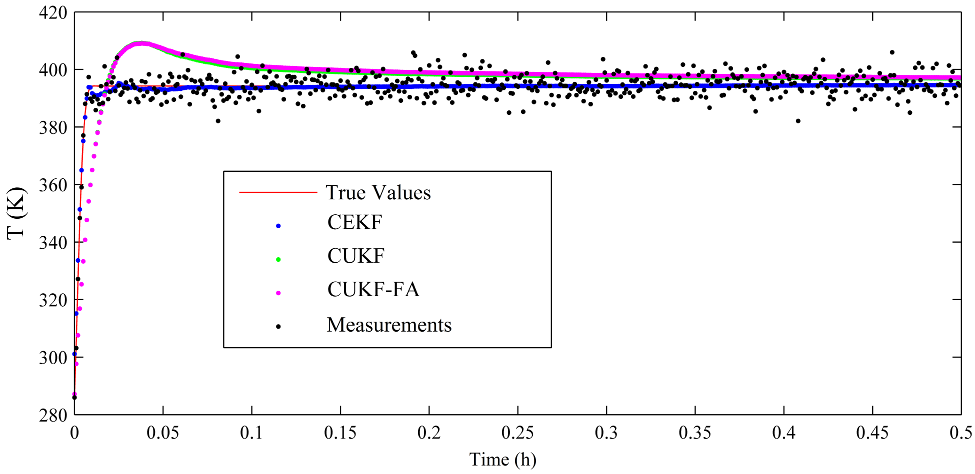

Figure 3.6: Temperature under joint estimation using the constrained filters.

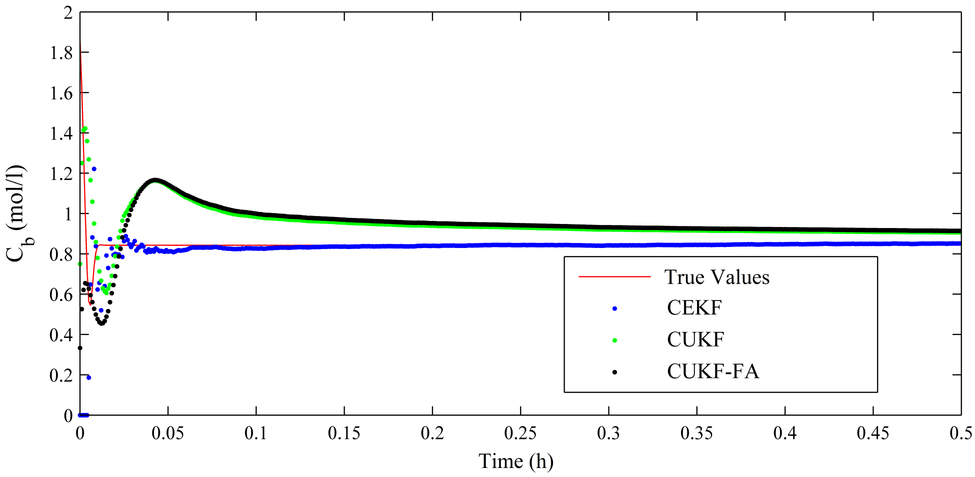

Figure 3.7: C_B composition under joint estimation using the constrained filters. .

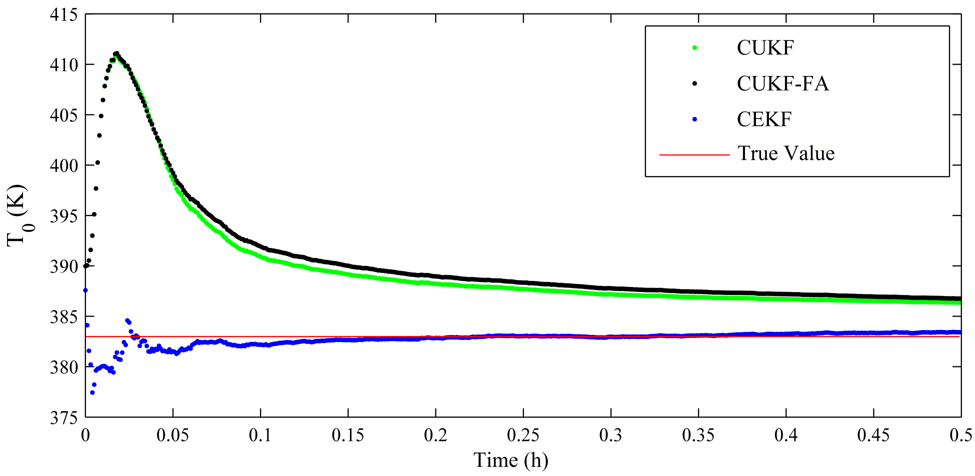

Figure 3.8: Parameter estimation (feed temperature T_0)

The UKF filters performed worse than the CEKF, most likely due to lack of system excitation. This type of filter requires that the system covariance matrix be positive definite along estimation, so that the Cholesky decomposition is possible. The sigma points must be chosen in such a way that if the decomposition is of the type $\mathbf{A}^T\mathbf{A}$, the matrix lines must be used, if it is of type $\mathbf{A}\mathbf{A}^T$ its columns form the set of sigma points. As the system excitation is reduced, it is possible that the covariance matrix eigenvalues approach zero and this causes a numerical error in Cholesky decomposition algorithm, which requires the decomposed matrix to be positive definite. In Figure 3.9 a typical dynamic eigenvalue behavior for the covariance matrix is shown.

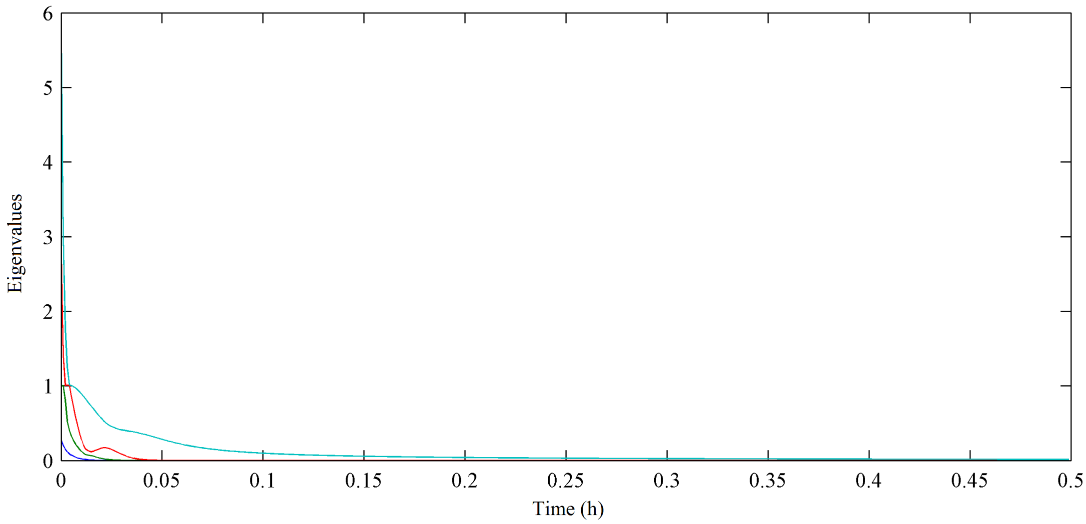

Figure 3.9: Eigenvalues dynamic behavior for the CUKF

For the algorithm robustness improvement, it is proposed the use of LDL^T decomposition, such that:

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T \tag{3.11}$$

The sigma points can be taken as the square root of $\mathbf{D}$ multiplied by $\mathbf{L}$:

$$\chi = \mathbf{L} \cdot \sqrt{\mathbf{D}} \tag{3.12}$$

It is observed that in this method the elements of $\mathbf{D}$ can be zero, unlike the Cholesky decomposition. If a element of this matrix is zeroed a sigma point is automatically lost due to the lack of excitation. However, since the unscented mean must be unchanged, this value is assigned as the mean of the remaining points, converging the algorithm to a steady value. Finally, the results obtained using this decomposition are the same as those generated using Cholesky decomposition.

However, the number of algorithm failures is reduced in the absence of excitation, leading to a more reliable estimation. The simulations in this chapter were made using both approaches, revealing a lower number of failures for the proposed method.

3.3 Predictive control routine implementations

By using a generic nonlinear model in a predictive controller, the resulting optimization problem has a computational complexity far superior to the constrained linear problem, which can be solved through quadratic programming (QP), or has analytical solution in absence of constraints. In Figure 3.10, a summary of the methods for solving the optimization problem resulting from MPC is showed.

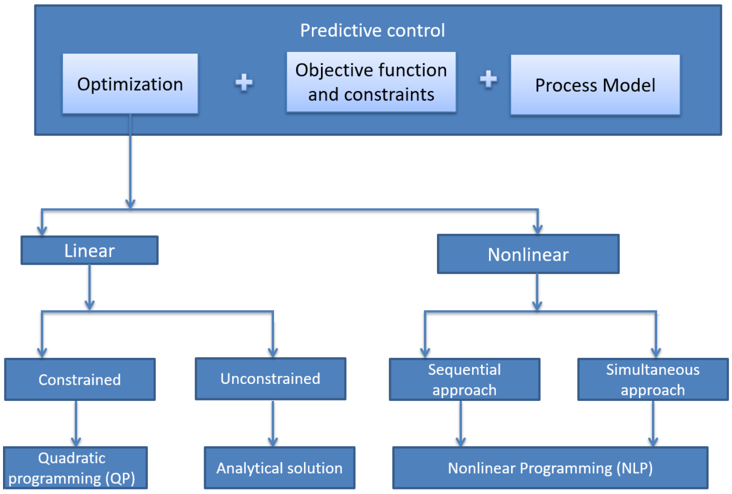

Figure 3.10: Numerical approaches for the MPC problem solution

For the non-linear problem, two approaches are used: the sequential and the simultaneous method (BIEGLER and HUGHES, 1985). MANENTI (2011) points out that both procedures have advantages and disadvantages. In the sequential method the dynamic model is treated separately from the optimization routine, in this way the optimization and the integration of the dynamic system are performed sequentially. This approach can be considered as partially parametrized, since only the control actions (optimization variables) are discretized. In addition:

- the optimal control problem size is decreased;
- an easier implementation is obtained;
- the algorithm optimal solution is always feasible (feasible path optimization).

In the simultaneous method, in addition to the control actions, the dynamic model is discretized and the resulting equations are imposed as equality constraints to the optimization problem. In this method, the optimization and integration converge simultaneously. The main simultaneous method characteristics are:

- It results in a sparse system, which can be solved efficiently;
- a feasible solution for the integration is not necessary for each optimization iteration, only in the final iteration it is mandatory.

Moreover, the simultaneous method increases the problem dimension. The computational cost of using each approach can vary in several aspects, such as system size and stiffness for system integration. TONEL (2008) has done a comparison between the simultaneous and sequential methods. In this study, three strategies were evaluated, two sequential approaches, using the integrators ode45 (MATLAB, 2008) and DASSLC (SECCHI, 1992–2007), and a simultaneous strategy using a discretized model through orthogonal collocation. The results showed that the sequential method using the DASSLC integrator was more efficient. Therefore, this approach was also used in this work; the optimizer uses the interior point algorithm that is implemented in the MATLAB software *fmincon* routine (MATLAB, 2008).

3.3.1 Code verification

In order to validate the implemented code for the non-linear controller, the control problem of a linear system was solved. The solution was compared to the commercial implementation of MATLAB (2008) predictive control toolbox. The linear system used is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.13}$$

The matrices were:

$$\mathbf{A} = \begin{bmatrix} -0,5 & 0,05 \\ 0,1 & -1 \end{bmatrix} \tag{3.14}$$

$$\mathbf{B} = \begin{bmatrix} -0,05 & 0,05 \\ 0,01 & -0.1 \end{bmatrix} \tag{3.15}$$

The control objective was to track x_1 and x_2 in the desired setpoint: $\mathbf{x}_{sp} = [3 \;\; 1]^T$. The same tuning was applied to both controllers.

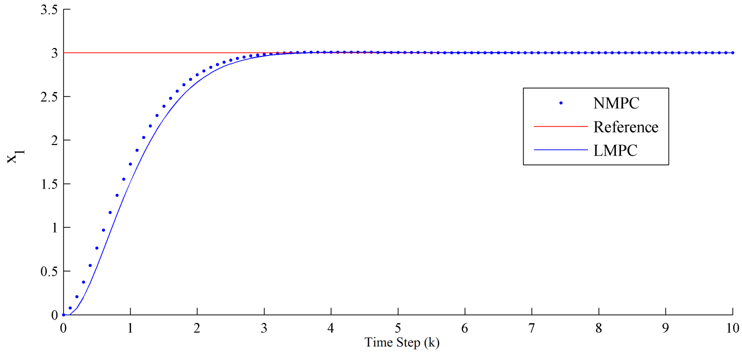

Figure 3.11: Controlled variable (x_1) for the MATLAB (2008) control toolbox and the implemented NMPC routine.

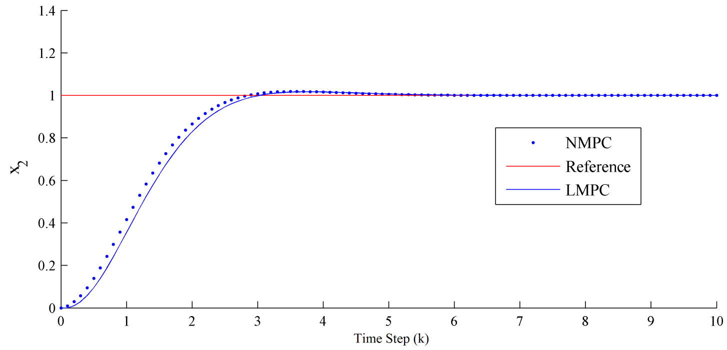

Figure 3.12: Controlled variable (x_2) for the MATLAB (2008) control toolbox and the implemented NMPC routine.

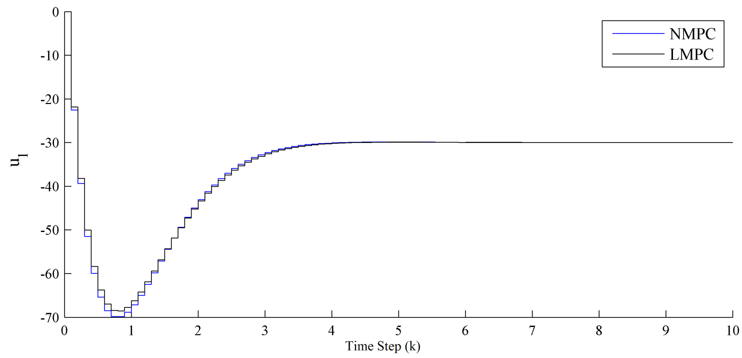

Figure 3.13: Manipulated variable (u_1) for the MATLAB (2008) control toolbox and the implemented NMPC routine.

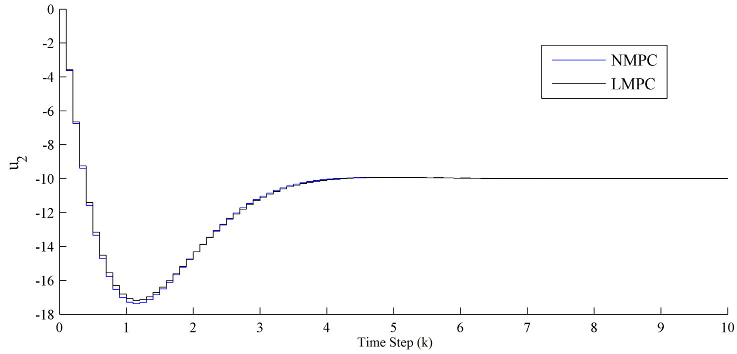

Figure 3.14: Manipulated variable (u_1) for the MATLAB (2008) control toolbox and the implemented NMPC routine.

Through the Figures 3.13 and 3.14 analysis, it can be noted that the optimal values are similar, resulting in a very similar state tracking (Figures 3.11 and 3.12). The small differences are explained by the intrinsic differences between the algorithms, such as the predictions that in nonlinear case are done by the DASSLC routine and in linear MPC the analytical solution is used. Moreover, the MATLAB toolbox does not allow the user to modify the estimator preferences in the same way that is provided by open routines. Finally, the nonlinear optimization routine (interior point) may find slightly different optimal values in comparison with the quadratic programming. Therefore, as the differences were small, the NMPC routine was considered validated and used in all simulations in this thesis.

3.4 Nonlinear control simulations

For the nonlinear test, the jacket balance was inserted in the van de Vusse reactor model, as suggested by KLATT and ENGELL (1998):

$$\frac{dT_k}{dt} = \frac{1}{m_k C_{pk}}[Q_k + K_w A_r (T - T_k)] \tag{3.16}$$

In this control scheme, it is considered the heat added to the reactor jacket Q_k as a manipulated variable. It was established as a control objective the composition of component B tracking and a zone control for reactor temperature. The setpoints for the composition controller have been modified over simulation time:

$$
\begin{aligned}
C_b &= 0,9 \ \ if \ \ 0 < k < 50 \\
C_b &= 1 \ \ if \ \ 50 \geq k < 140 \\
C_b &= 1,15 \ \ if \ \ k \geq 140
\end{aligned}
\tag{3.17}
$$

It is pointed out that the third setpoint is unreachable, as shown in Figure 3.1, leading to instability if a linear controller was used due to gain sign inversion. The reactor temperature along the simulation was kept into the zone:

$$
406,15 < T < 412,15 \tag{3.18}
$$

The second manipulated variable is the reactor volumetric feed, which modifies the dilution rate and, consequently, the residence time. Therefore, the system has the configuration showed in Table 3.1:

Table 3.1: Controled and manipulated variables for the NMPC problem

Variable	Description	Function
F	Feed flow rate (L/h)	manipulated
Q_k	Heat exchanged by the jacket (kJ/h)	manipulated
C_b	Desired product concentration	Setpoint control
T	Reactor temperature (K)	Zone control

3.4.1 Nominal case

For the first simulations, the nominal case was used for further comparison with the upcoming cases and for controller tuning. In this case, the simulated system has the following characteristics:

- the plant and the controller employ the same model;
- the states are perfectly measured.

The controlled variable results C_b (Figure 3.15) show the first two reachable setpoints and the final unreachable setpoint. As noticed, the first two setpoints are offset free, while the third one is stabilized in its maximum, below the setpoint. In Figure 3.16 the temperature dynamics is shown, it remains constrained in the zone. The manipulated variables (Figures 3.17 and 3.18) do not reach their maximum and stabilize at the unreachable setpoint.

It is observed that, for the two initial setpoint changes, the cooling / heating jacket is almost not used. In the third change, once the predictive controller realizes that only the flow modification is not enough, the jacket heat is used to decrease the steady state error. It is observed that this system does not satisfy the offset elimination conditions, since there is no set of manipulated variables that could generate the setpoint value for the nominal case (MORARI and MAEDER, 2012).

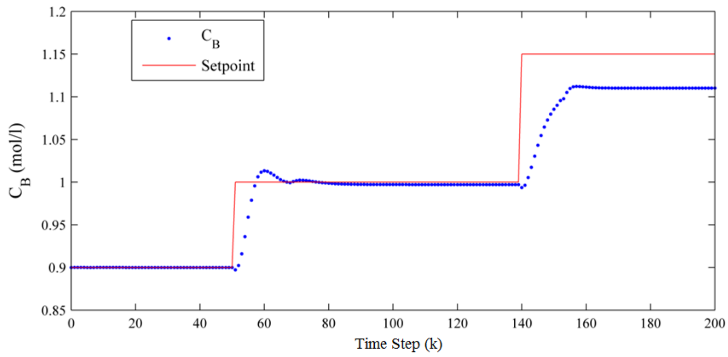

Figure 3.15: C_b for nominal control

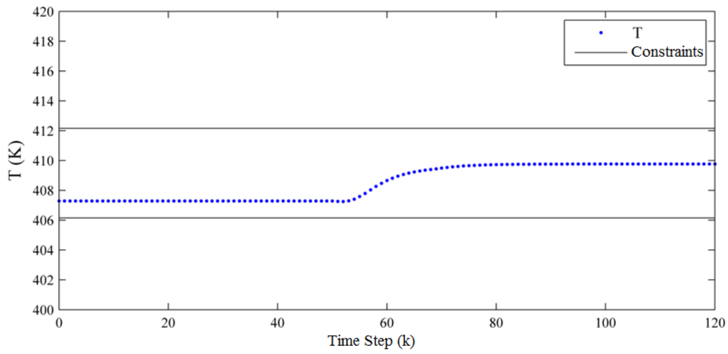

Figure 3.16: Reactor temperature (T) for nominal zone control

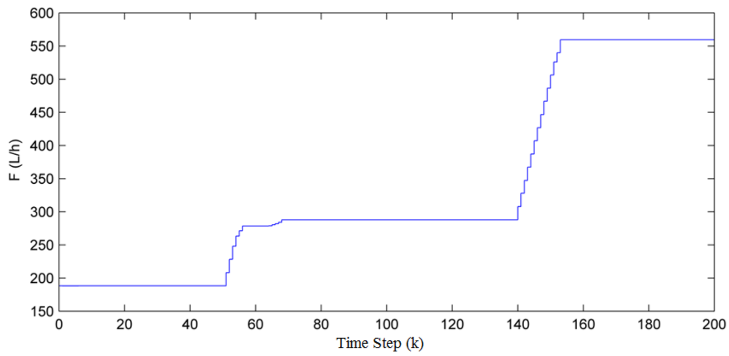

Figure 3.17: Feed flow rate (F) for nominal control

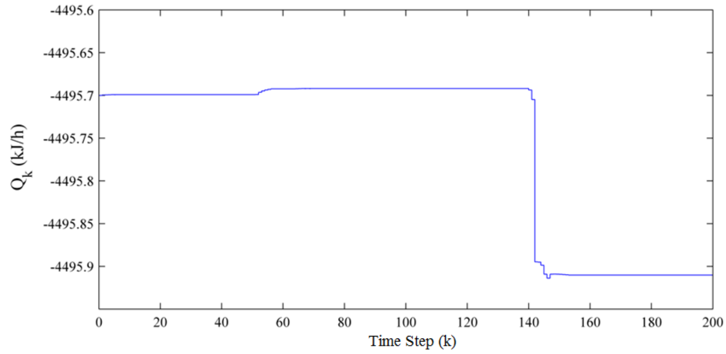

Figure 3.18: Jacket heat (Q_k) for nominal control

3.5 Control under noisy partial measurements

The estimator test is fundamental for the proper controller performance. In this simulation it was considered that the state vector was partially measured; only the reactor temperature was considered monitored by a physical sensor. The measurements were corrupted by a Gaussian noise with standard deviation of 1% of the nominal value. The other variables were estimated using the constrained Kalman filter (CEKF), which has obtained the best performance in the analysis of the estimators (Section 2.2). In Figure 3.19, it is possible to observe the effect that the state observer exerts on the noisy measurements, being able to reconcile the data. Although there are measured values outside the constraints, the reconciled values remain within the constraints. The EKF filter estimation, without constraints, was placed on the figure to show that this is not due to the constraints imposed on the state estimation.

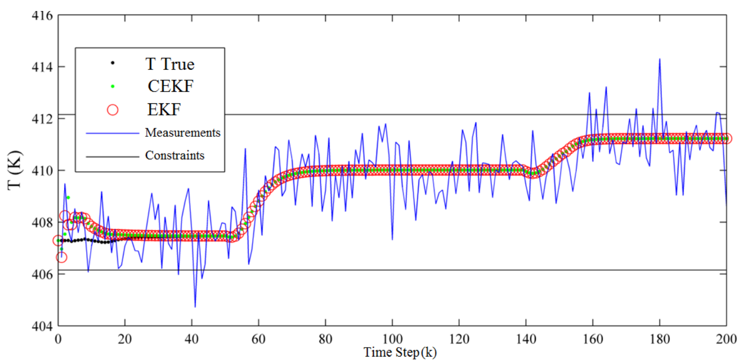

Figure 3.19: Temperature control for noisy measurements

The control of component B composition, as displayed in Figure 3.20, reveal a offset due to noise in the second setpoint change. It is important to highlight that, for this setpoint value, a tuning that produces an offset free controller is possible, however, the nominal tuning was used for comparison purposes.

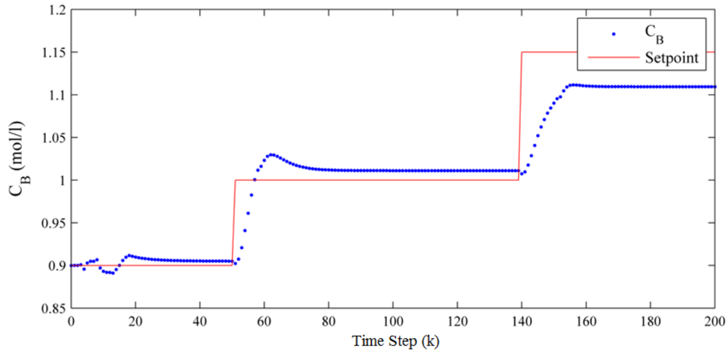

Figure 3.20: C_b for noisy measurements and partially measured state vector

The results for the manipulated variables exhibit significant differences between the nominal and the noisy case, as shown in Figures 3.21 e 3.22. In the noisy case, the feed flow rate is more oscillatory, moreover, the jacket heat is overly used, resulting in energy waste. Despite this problems, the NMPC controller is able to stabilize the system, including the unreachable setpoint.

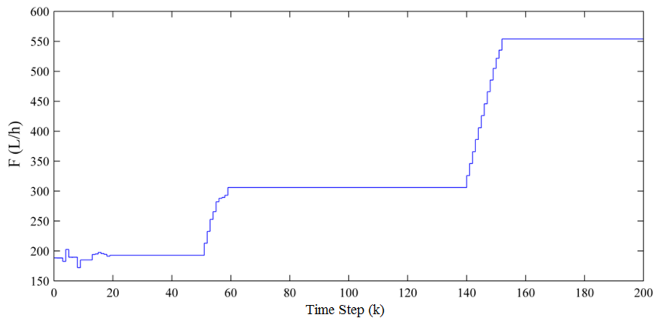

Figure 3.21: F for noisy measurements and partially measured state vector

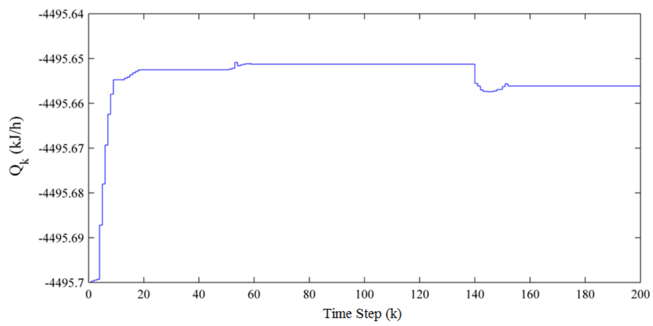

Figure 3.22: Q_k for noisy measurements and partially measured state vector

3.6 Predictive control with online parameter estimation using observers

Usually, the predictive control algorithm feedback is based on a disturbance model, such as in the classic DMC approach. In this section, a comparison between this standard approach and the adaptive MPC strategy was carried out. The disturbance model is given by (MACIEJOWSKY, 2000):

$$\mathbf{d}(k) = \mathbf{y}(k) - \mathbf{y}_p(k|k-1) \tag{3.19}$$

The disturbance vector is of dimension two, equivalent to the controlled variables. For the ANMPC controller, the feed temperature is considered an unknown parameter to be estimated. This parameter is inserted into the state vector, which is estimated by a Kalman observer. Since there is no dynamic model for this parameter, the standard assumption of slow variation was made. The disturbance inserted in this variable, for tests purposes, does not satisfy this assumption, as showed in Figure 3.23.

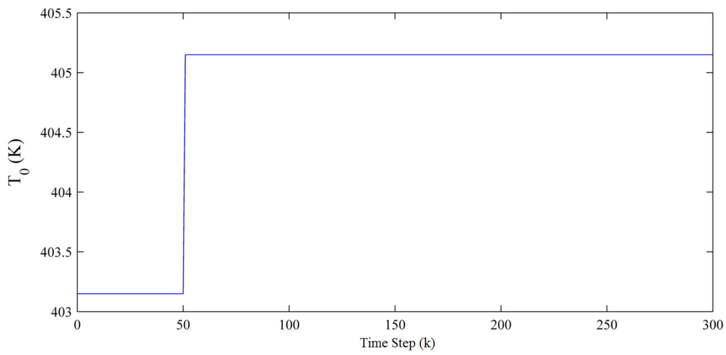

Figure 3.23: Disturbance in the feed temperature

In Figure 3.24, the dynamic temperature evolution is shown. The controllers were tuned using the same parameters. As expected, both controllers are slower than the nominal case. The controller using the additive disturbance model, however, presented oscillations that are not noticed in the adaptive controller. Therefore, the disturbance model has inserted instability in the system, being not able to reject properly the feed temperature disturbance.

By analyzing the reactor temperature (Figure 3.25), the results reveal that the constraints are violated when the model for the disturbance is used. At the same time, the ANMPC controller was able to maintain the temperature within the constraints. The model used in the ANMPC provides a better disturbance rejection,

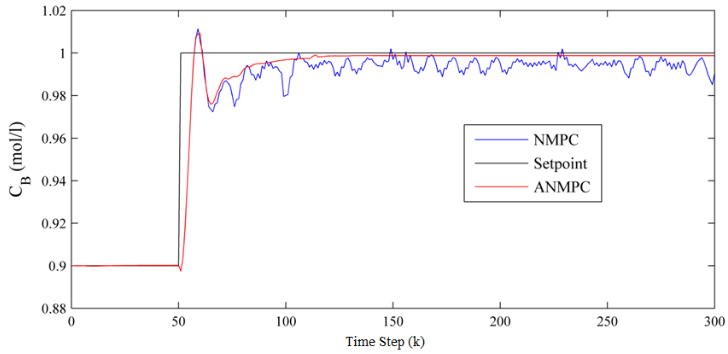

Figure 3.24: C_b tracking for the ANMPC and NMPC controllers

being able to correct the future trajectory avoiding constraint violation. On the other hand, the disturbance model is not able to stabilize the system, inserting periodic oscillations. We emphasize that both systems have used the tuning obtained for a nominal situation.

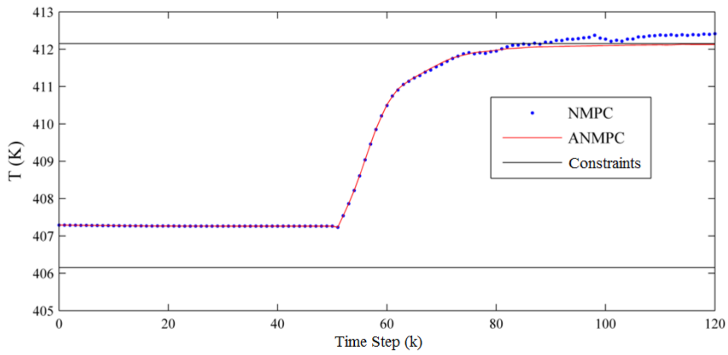

Figure 3.25: Temperature (T) zone control for the NMPC and ANMPC controllers

The manipulated variables also present quite different behaviors. The feed flowrate is manipulated in the same way for the two controllers at the initial time-steps, however, as the simulation evolves, the ANMPC controller is able to stabilize the system, while the NMPC continues to oscillate, as seen in Figure 3.26. In Figure 3.27, the power supplied to the cooling jacket over time is shown. It can be seen that the controller with the disturbance model acts in the opposite direction when the temperature of the inlet jacket is modified, progressing without stabilizing until the end of the simulation. The updated model, on the other hand, stabilizes the manipulated variable.

In Figure 3.28 the closed-loop estimation result of the input temperature using the constrained Kalman filter is shown. Note that the convergence to the true value is fast, taking approximately 50 time-steps to recover the true inlet temperature value.

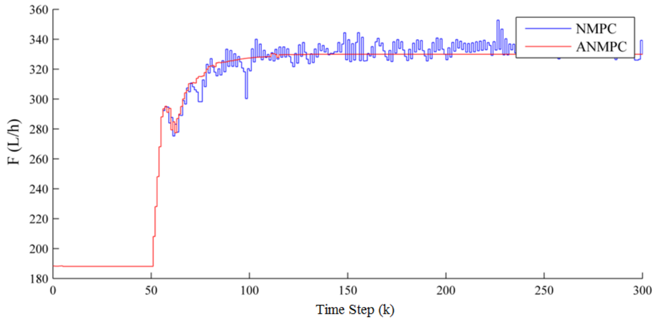

Figure 3.26: Feed flow rate (F) for the ANMPC and NMPC controllers

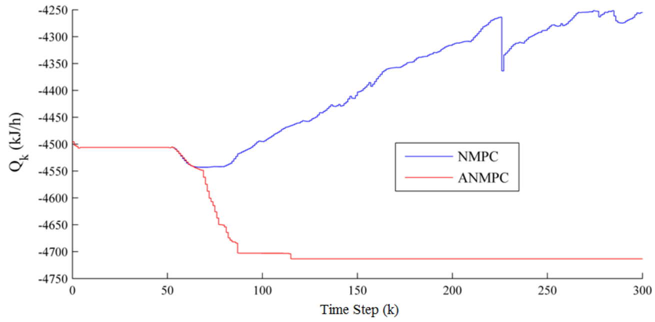

Figure 3.27: Jacket power (Q_k) for the ANMPC and NMPC controllers

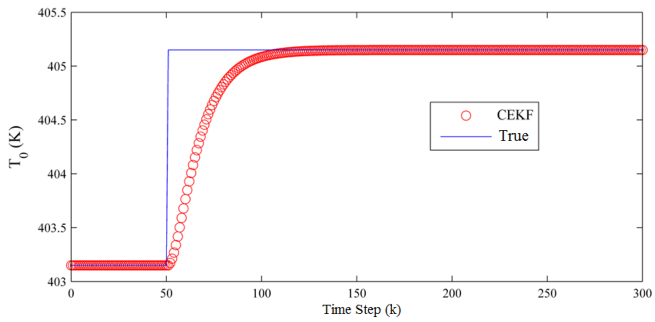

Figure 3.28: Parameter estimation (T_0) using the constrained Kalman filter (CEKF)

3.7 Conclusions

In this chapter we presented an analysis of the current algorithms for state estimation aiming joint estimation and use in model predictive control. Results show that the constrained Kalman filter, using an analytical jacobian is a great option for joint estimation. Additionally, a framework for numerical solving the MPC problem under parameter estimation was developed. Moreover, the feedback based on model parameter updating may have a better performance in comparison with the standard additive disturbance approach. We highlight that both methods are using assumptions that are not valid, however, the additive disturbance (a pure output method) appears to not be able to generate properly predictions. In addition, only one parameter was estimated and two disturbances were inserted in order to track the feed temperature disturbance. This results show the great impact that the use of parameter estimation as a pure-input non additive disturbance rejection method may have in the control performance.

Despite the large application of Kalman filters, for the nonlinear problem, the use of parameter estimation using this approach in a robust MPC is difficult. This algorithm presents some challenges for future error estimation and then guarantee a robust region for the estimates. An alternative for this class of estimators is the interval observers. In the following chapter, we present a strategy using this approach in order to obtain a robust MPC under state feedback.

Chapter 4

Robust Discrete-time Set-Based Adaptive Predictive Control for Nonlinear System[1]

4.1 Introduction

Model Predictive Control (MPC) has experienced considerable attention in the academic literature in the last two decades. Moreover, it has also been largely applied by industries, due to its ability to enforce constraints and handle multivariable process effectively and efficiently. However, the presence of uncertainty in the MPC problem formulation remains a challenging topic. The presence of uncertainty requires feedback and optimization over a sequence of control laws rather than optimization over sequences of control actions, as in nominal MPC MAYNE (2014). Despite academic effort in the design of robust nonlinear model predictive control (NMPC) systems, the problems associated with parametric uncertainties remains a considerable challenge in applications. The presence of parametric uncertainties can have severe implications in the implementation of reliable NMPC systems. In classical control, this task can be handled using a vast array of adaptive control and adaptive estimation techniques. The situation in MPC poses some additional challenges. The main problem with the application of an adaptive control approach in NMPC systems is that the uncertain parameters may impact the quality of the model predictions drastically and, hence, the performance of the control system. It is therefore imperative that the NMPC approach preserves robustness to parametric uncertainties while taking full advantage of the potential NMPC performance gains.

[1]A version of this chapter was published in the Journal of Process Control (GONÇALVES and GUAY, 2016)

The problem of using measurements for online update of model parameters in MPC (so called adaptive MPC) has received some attention in the literature. In general, the model and the uncertainty descriptions of a robust MPC are configured for a nominal set of operating conditions that are typically not updated. The nominal parameter uncertainties are thus lumped with other structured or unstructured uncertainty descriptions which yields conservative robust control systems. Parametric uncertainty is usually handled by imposing bounds on the unknown parameters. Various robust MPC mechanism can then be employed to mitigate their impact on the model predictions and the MPC system. Min-max robust MPC approaches can be used to handle such parametric uncertainties (see RAIMONDO *et al.* (2009) and the references therein). The conservatism associated with such approaches can be overcome if one is able to anticipate the effect of future changes in the uncertainties. In the case of adaptive MPC, the objective is to use online learning algorithms that use plant measurements to update the uncertain parameters. Such algorithms are usually equipped with some guarantees of convergence of the parameters in a way that can be used to forecast their impact on future model predictions. For linear systems some results are available. In FUKUSHIMA *et al.* (2007), an adaptive approach is considered where an exponential decay for parametric uncertainty is used in the model prediction. An interesting result was proposed in MARAFIOTI *et al.* (2014) where a persistency of excitation condition is used to prove that robust feasibility is preserved if no states constraints are used.

In a recent study TANASKOVIC *et al.* (2013), a set-membership adaptive MPC approach was proposed. In this technique, a class of linear systems represented by impulse response coefficients convolution models is considered. Under the assumption that the model of the uncertain plant belongs to a class of linear systems with bounded impulse response coefficients, a set-membership update strategy is used to identify models that are consistent with past input-output responses of the uncertain linear system. The technique is shown to provide accurate set membership assignment and guarantees robust performance of the unknown linear systems. In the context of the current study, the approach proposes to update the parameters following a model set-membership approach following an empirical model approach. Furthermore, it is limited to open-loop stable systems, with the possibility of integrating behaviour. A similar approach is proposed in CANALE *et al.* (2013) where a set membership identification technique due to MILANESE and NOVARA (2004) is used to identify nonlinear systems from a class of Lipschitz nonlinear operators based on the closeness of the process data. Again, the approach is limited to open-loop stable systems. The current study provides a robust adaptive stabilization result for a class of uncertain nonlinear systems. The parameterization is assumed to be known but the uncertainty in the parameters can be effectively updated in real-time to minimize the impact of the uncertain parameters.

For the adaptive MPC control of nonlinear systems, some authors have proposed the use of adaptive neural network models (for example see ALEXANDRIDIS and SARIMVEIS (2005), (AKPAN and HASSAPIS, 2011) and SALAHSHOOR *et al.* (2013)). For nonlinear constrained systems, ADETOLA *et al.* (2009) proposed a robust framework for continuous time systems, in which the transient effect of parameter estimation error was explicitly used in the robust control problem. In ADETOLA and GUAY (2011), the previous work was extended for continuous systems with disturbances. Finally, an adaptive robust economic MPC, based on the results found in ADETOLA *et al.* (2009) and ADETOLA and GUAY (2011), was proposed in GUAY and ADETOLA (2013).

Another framework to design an adaptive predictive controller considers the use of Kalman Filters and Moving Horizon Estimation (MHE) to obtain estimates of the parameters. In this approach, the state vector is augmented with the unknown parameter values under the assumption of a constant parameter vector SIMON (2006). In the context of nonlinear adaptive MPC, this technique was used by FINKLER *et al.* (2014) for joint estimation using an Extended Kalman Filter (EKF) for a polymerization reactor. A combination of MHE and MPC in the adaptive framework is shown in CHEN *et al.* (2012), where the optimal dosing of cancer chemotherapy problem is addressed. This approach can be applied to a large class of nonlinear systems. However, the dynamic uncertainty estimation for application in a robust MPC problem is not a trivial problem. Moreover, the computation of the parameter estimates using an MHE approach increases the computational cost, since two nonlinear optimization problems should be solved sequentially. The presented solution proposes the use of an algorithm that allows the dynamic uncertainty set update and preserves the computational cost of usual nonlinear MPC, leading to an implementable robust algorithm.

In this work, we establish a theoretical basis for the analysis of robust adaptive MPC control system subject to exogenous disturbances for a class of discrete-time nonlinear control systems. The result generalizes the continuous-time approach first proposed in ADETOLA *et al.* (2009). No claims are made concerning the computational requirements of the proposed min-max approach to the adaptive MPC technique. However, it is argued that a Lipschitz-based approach provides a conservative approximation of the min-max approach that retains all of the stability and robustness properties. The uncertainties associated with the parameters are handled using a new set-based estimation approach for a class of nonlinear discrete-time systems that guarantees contraction of the uncertainty set in the presence of a persistency of excitation condition. Moreover, it is shown how this set-based approach can be formulated in the context of nonlinear adaptive MPC approach for discrete-time systems in the presence of parameter uncertainties and exogenous disturbances.

The remainder of the chapter is structured as follows. The problem description is given in Section 4.2. The parameter estimation routine is presented in Section 4.3. Two approaches to robust adaptive model predictive control are detailed in Section 4.4. This is followed by a simulation example in Section 4.6 and brief conclusions in Section 4.7.

4.2 Problem Description

Consider the uncertain discrete-time nonlinear system[2]:

$$x_{k+1} = x_k + F(x_k, u_k) + G(x_k, u_k)\theta + \vartheta_k \triangleq \mathcal{F}(x_k, u_k, \theta, \vartheta_k) \tag{4.1}$$

where the disturbance $\vartheta_k \in \mathcal{D} \subset \mathbb{R}^{n_d}$ is assumed to satisfy a known upper bound $\|\vartheta_k\| \leq M_\vartheta < \infty$. The objective of the study is to (robustly) stabilize the plant to some target set $\Xi \subset \mathbb{R}^{n_x}$ while satisfying the point-wise constraints $x_k \in \mathcal{X} \in \mathbb{R}^{n_x}$ and $u_k \in \mathcal{U} \in \mathbb{R}^{n_u}$, $\forall k \in \mathbb{Z}$. The target set is a compact set, contains the origin and is robustly invariant under no control. It is assumed that θ is uniquely identifiable and lies within an initially known compact set $\Theta^0 = B(\theta_0, z_\theta)$ where θ_0 is a nominal parameter value, z_θ is the radius of the parameter uncertainty set.

Remark 1 *In this study, the exogenous variable ϑ_k represents an unstructured bounded time-varying uncertainty. We do not provide any additional structure, such as a state dependent disturbance matrix, since this is assumed to be expressed by the term $G(x_k, u_k)\theta$ in* (4.1).

4.3 Parameter and Uncertainty Set Estimation

In this section, we present and analyze the proposed set-based parameter estimation technique.

4.3.1 Parameter Adaptation

The preferred parameter estimation technique is first presented. The main idea behind the proposed technique is the definition of an implicit regression model. The

[2]We use the notation x_k for the variables of the state-feedback problem and $x(k)$ for the output-feedback and estimation problem in the next chapter

implicit model is based on the definition of a vector of auxiliary variables, denoted by η_k. The dynamics of η_k forms the basis for the proposed estimation approach.

The first element required is filtered form of the regressor vector $G(x_k, u_k)$ denoted by ω_k. The vector ω_k is obtained using the following recursion:

$$\omega_{k+1} = \omega_k + G(x_k, u_k) - K_k \omega_k, \qquad \omega_0 = 0 \tag{4.2}$$

where K_k is a correction factor at time step k. Note that ω_k has the same dimension as $G(x_k, u_k)$

We let $\hat{\theta}_k$ be the vector of parameter estimates at time step k. Using the process model and the filter (4.2), we propose the following state predictor:

$$\begin{aligned} \hat{x}_{k+1} &= \hat{x}_k + F(x_k, u_k) + G(x_k, u_k)\hat{\theta}_{k+1} + K_k e_k - \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) \\ &+ K_k \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) \end{aligned} \tag{4.3}$$

where $e_k = x_k - \hat{x}_k$ is the state estimation error at time step k. The state predictor is used to generate information about the parameter estimates.

We then let $\tilde{\theta}_k = \theta - \hat{\theta}_k$ denote that parameter estimation error. Using the state predictor (4.3) and the filter (4.2), the prediction error dynamics are given by:

$$\begin{aligned} e_{k+1} &= e_k + G(x_k, u_k)\tilde{\theta}_{k+1} - K_k e_k + \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) \\ &- K_k \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) + \vartheta_k \\ e_0 &= x_0 - \hat{x}_0. \end{aligned} \tag{4.4}$$

We can now define the auxiliary variable η_k as

$$\eta_k = e_k - \omega_k \tilde{\theta}_k. \tag{4.5}$$

The insight behind the use of the variable η_k is twofold. First, the vanishing of this variable implies $e_k = \omega_k \tilde{\theta}_k$. Thus, the vanishing of η_k and $\tilde{\theta}_k$ implies that the parameter estimation has been solved. The second, and most important, aspect is that the choice of parameter update law is clearly independent of the error dynamics.

Upon substitution of (4.2) and (4.4), the auxiliary variable dynamics are given by:

$$\begin{aligned} \eta_{k+1} &= \eta_k - K_k \eta_k + \vartheta_k \\ \eta_0 &= e_0 \end{aligned} \tag{4.6}$$

Since ϑ_k is unknown, it is necessary to use an estimate, $\hat{\eta}$, of η. This is done to filter its impact on the parameter estimation scheme. The estimate, $\hat{\eta}_k$, is generated by the recursion:

$$\hat{\eta}_{k+1} = \hat{\eta}_k - K_k \hat{\eta}_k \tag{4.7}$$

The resulting dynamics of the η estimation error, $\tilde{\eta}_k = \eta_k - \hat{\eta}_k$ are:

$$\tilde{\eta}_{k+1} = \tilde{\eta}_k - K_k \tilde{\eta}_k + \vartheta_k \tag{4.8}$$

We are now ready to present the proposed parameter estimation scheme. We first propose the n_p by n_p identifier matrix Σ_k with dynamics governed by the following matrix recursion:

$$\Sigma_{k+1} = \Sigma_k + \omega_k^T \omega_k, \quad \Sigma_0 = \alpha I \succ 0 \tag{4.9}$$

where α is a positive constant to be assigned.

Using standard argument, its inverse is generated by the following recursion:

$$\Sigma_{k+1}^{-1} = \Sigma_k^{-1} - \Sigma_k^{-1^T} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} \omega_k \Sigma_k^{-1}, \quad \Sigma_0^{-1} = \frac{1}{\alpha} I \succ 0. \tag{4.10}$$

From (4.3), (4.2), and (4.7), the parameter update law is defined as follows:

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} (e_k - \hat{\eta}_k) \tag{4.11}$$

To ensure that the parameter estimates remain within the constraint set Θ_k, we propose to use a projection operator of the form:

$$\bar{\hat{\theta}}_{k+1} = \text{Proj}\{\hat{\theta}_k + \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} (e_k - \hat{\eta}_k), \Theta_k\} \tag{4.12}$$

The operator $Proj$ represents an orthogonal projection onto the surface of the uncertainty set applied to the parameter estimate. The parameter uncertainty set is defined by the ball function $B(\hat{\theta}_c, z_{\hat{\theta}c})$, where $\hat{\theta}_c$ and $z_{\hat{\theta}c}$ are the parameter estimate vector and set radius obtained at the latest set update.

Following GOODWIN and SIN (1984), the projection operator is designed such that:

- $\hat{\theta}_{k+1} \in \Theta_k$
- $\bar{\tilde{\theta}}_{k+1}^T \Sigma_{k+1} \bar{\tilde{\theta}}_{k+1} \leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1}$

In the remainder of this section, it is shown that the parameter update law (4.11) guarantees convergence of parameter estimates to the true values. The result is stated as a series of two lemmas, Lemma 2 and 3.

Lemma 2 *HADDAD and VIJAYSEKHAR (2008) Consider the system*

$$x_{k+1} = Ax_k + Bu_k \tag{4.13}$$

where A is a stable matrix with eigenvalues inside the unit circle and B is a matrix of appropriate dimension. Then, it can be shown that

$$\sum_{k=0}^{K-1} x_{k+1}^T x_{k+1} \leq \delta^2 \sum_{k=0}^{K-1} u_k^T u_k \tag{4.14}$$

for some $\delta > 0$ and $K-1 > 0$.

Let l_2 denote the space of square finitely summable signals and consider the following lemma.

Lemma 3 *The identifier (4.9) and parameter update law (4.11) are such that $\tilde{\theta}_k = \theta_k - \hat{\theta}_k$ is bounded. Furthermore, if*

$$\vartheta_k \in l_2 \quad or \quad \sum_{k=0}^{\infty} [\|\tilde{\eta}_k\|^2 - \gamma \|e_k - \hat{\eta}_k\|^2] < +\infty \tag{4.15}$$

and

$$\lim_{k\to\infty} \Sigma_k = \infty \tag{4.16}$$

are satisfied, then $\tilde{\theta}_k$ converges to 0 asymptotically.

Proof: Let $V_{\tilde{\theta}k} = \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k$ It follows from the properties of the projection operator that:

$$V_{\tilde{\theta}k+1} - V_{\tilde{\theta}k} \;=\; \bar{\tilde{\theta}}_{k+1}^T \Sigma_{k+1} \bar{\tilde{\theta}}_{k+1} - \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k \leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1} - \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k.$$

Using the parameter update law, one can write $\tilde{\theta}_{k+1}$ as:

$$\begin{aligned}
\tilde{\theta}_{k+1} &= \tilde{\theta}_k - \Sigma_k^{-1}\omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k) \\
&= \tilde{\theta}_k - \Sigma_k^{-1}\omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (w_k \tilde{\theta}_k + \tilde{\eta}_k)
\end{aligned}$$

or,

$$\tilde{\theta}_{k+1} = \Sigma_{k+1}^{-1}\Sigma_k \tilde{\theta}_k - \Sigma_k^{-1}\omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \tilde{\eta}_k. \tag{4.17}$$

Upon substitution of the parameter update law, the identifier matrix dynamics, the filter dynamics and the auxiliary variable dynamics, the rate change of the $V_{\tilde{\theta}k}$ is given by:

$$\begin{aligned}
V_{\tilde{\theta}k+1} - V_{\tilde{\theta}k} \leq& -(e_k - \hat{\eta}_k)^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k) \\
&+ \tilde{\eta}_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \tilde{\eta}_k
\end{aligned} \tag{4.18}$$

From the $\tilde{\eta}_k$ dynamics given in (4.8), it follows from Lemma 2 if $\vartheta_k \in l_2$ then $\tilde{\eta}_k \in l_2$. Taking the limit as $k \to \infty$, the inequality becomes

$$\begin{aligned}
\lim_{k\to\infty} V_{\tilde{\theta}k} &= V_{\tilde{\theta}0} + \sum_{k=0}^{\infty} V_{\tilde{\theta}k+1} - V_{\tilde{\theta}k} && (4.19)\\
&\leq V_{\tilde{\theta}0} - \sum_{k=0}^{\infty} \left[(e_k - \hat{\eta}_k)^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k)\right] && (4.20)\\
&\quad + \sum_{k=0}^{\infty} \left[\tilde{\eta}_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \tilde{\eta}_k\right] && (4.21)
\end{aligned}$$

By the boundedness of the trajectories of the system, it follows that there exists a number $\gamma > 0$ such that

$$1 \geq \left\| \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \right\| \geq \gamma.$$

as a result, one obtains the following inequality

$$\lim_{k\to\infty} V_{\tilde{\theta}k} \leq V_{\tilde{\theta}0} - \gamma \sum_{k=0}^{\infty} \left[(e_k - \hat{\eta}_k)^T (e_k - \hat{\eta}_k)\right] + \sum_{k=0}^{\infty} \left[\tilde{\eta}_k^T \tilde{\eta}_k\right] \qquad (4.22)$$

Therefore if the conditions (4.15) are met then the right hand side of (4.22) is finite. As a result, one concludes that

$$\lim_{k\to\infty} \tilde{\theta}_k = 0 \qquad (4.23)$$

as required. ■

4.3.2 Set Update

An update law that measures the worst-case progress of the parameter update law is adapted from the one proposed in ADETOLA and GUAY (2009):

$$\begin{aligned}
z_{\hat{\theta}k} &= \sqrt{\frac{V_{z\hat{\theta}k}}{4\lambda_{min}(\Sigma_k)}} && (4.24a)\\
V_{z\theta k+1} &= V_{z\hat{\theta}k} \\
&\quad - (e_k - \hat{\eta}_k)^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k) + (\frac{M_\vartheta}{K_k})^2 && (4.24b)\\
V_{z\hat{\theta}0} &= 4\lambda_{max}(\Sigma_0)(z_{\hat{\theta}0})^2 && (4.24c)
\end{aligned}$$

The parameter uncertainty set, defined by the ball function $B(\hat{\theta}_c, z_c)$ is updated using the parameter update law (4.11) and the error bound (4.24) according to the following algorithm:

Algorithm 1 *beginning at time step $k = 0$, the set is adapted according to the following iterative process*

1. **Initialize** $z_{\hat{\theta}c} = z_{\hat{\theta}0}, \hat{\theta}_c = \hat{\theta}_0$

2. *at time step k, using equations (4.11) and (4.24) perform the update*

$$(\hat{\theta}_c, z_{\hat{\theta}c}) = \begin{cases} (\hat{\theta}_k, z_{\hat{\theta}k}) & \text{if } \; z_{\hat{\theta}k} \leq z_c - \left\|\hat{\theta}_k - \hat{\theta}_c\right\| \\ (\hat{\theta}_c, z_{\hat{\theta}c}) & \text{otherwise} \end{cases} \tag{4.25}$$

3. *Return to step two and iterate, incrementing to time step $k+1$*

Lemma 4 *The algorithm ensures that*

1. *the set is only updated when updating will yield a contraction,*

2. *the dynamics of the set error bound described in (4.24) are such that they ensure the non-exclusion of the true value $\theta \in \Theta_k$, $\forall k$ if $\theta_0 \in \Theta_0$.*

Proof:

1. If $\Theta_{k+1} \nsubseteq \Theta_k$ then
$$\sup_{s \epsilon \Theta_{k+1}} \left\|s - \hat{\theta}_k\right\| \geq z_{\hat{\theta}k} \tag{4.26}$$

   However, it is guaranteed by the set update algorithm presented, that Θ, at update times, obeys the following

$$\begin{aligned} & \sup_{s \epsilon \Theta_{k+1}} \left\|s - \hat{\theta}_k\right\| \\ \leq \; & \sup_{s \epsilon \Theta_{k+1}} \left\|s - \hat{\theta}_{k+1}\right\| + \left\|\hat{\theta}_{k+1} - \hat{\theta}_k\right\| & (4.27) \\ \leq \; & z_{\hat{\theta}k+1} + \left\|\hat{\theta}_{k+1} - \hat{\theta}_k\right\| \leq z_{\hat{\theta}k} & (4.28) \end{aligned}$$

   This contradicts (4.26). Therefore, $\Theta_{k+1} \subseteq \Theta_k$ at time steps where Θ is updated.

2. It is known, by definition, that

$$V_{\tilde{\theta}0} \leq V_{z\theta 0}, \qquad \forall k \geq 0 \tag{4.29}$$

   Since, $V_{\tilde{\theta}k} = \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k$,

$$\left\|\tilde{\theta}_k\right\| \leq \frac{V_{z\hat{\theta}k}}{\lambda_{min}(\Sigma_k)} = 4z_{\hat{\theta}k}^2, \qquad \forall k \geq 0 \tag{4.30}$$

Therefore, if $\theta \epsilon \Theta_0$, then $\theta \epsilon \Theta_k \;\; \forall\, k \;\geq\; 0$.

■

4.4 Robust adaptive MPC

4.4.1 A Min-max Approach

The formulation of the min-max MPC consists of maximizing a cost function with respect to $\theta \in \Theta$, $\vartheta \in \mathcal{D}$ and minimizing over feedback control policies κ. This formulation is a simple application of the min-max approach proposed in the continuous-time setting.

The proposed robust receding horizon control law is given by:

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) \triangleq \kappa^*(0, x, \hat{\theta}, z_\theta) \tag{4.31a}$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot,\cdot,\cdot,\cdot)} J(x, \hat{\theta}, z_\theta, \kappa) \tag{4.31b}$$

where

$$J(x, \hat{\theta}, z_\theta, \kappa) \triangleq \max_{\theta \in \Theta,\ \vartheta_k \in \mathcal{D}} \sum_{k=0}^{T-1} L(x_k^p, u_k^p) d\tau + W(x_T^p, \tilde{\theta}_T^p) \tag{4.32a}$$

$$s.t.\ \ \forall k \in [0,\, T]$$

$$x_{k+1}^p = x_k + f(x_k^p, u_k^p) + g(x_k^p, u_k^p)\theta + \vartheta_k, \ \ x_0^p = x \tag{4.32b}$$

$$\hat{x}_{k+1}^p = \hat{x}_k^p + F(x_k^p, u_k^p) + G(x_k^p, u_k^p)\hat{\theta}_{k+1}^p \tag{4.32c}$$

$$+K_k e_k^p - \omega_k^p(\hat{\theta}_k^p - \hat{\theta}_{k+1}^p) + K_k \omega_k^p(\hat{\theta}_k^p - \hat{\theta}_{k+1}^p) \tag{4.32d}$$

$$w_{k+1}^p = w_k + G(x_k^p, u_k^p) - k_w\, w_k^p, \ \ w_0^p = w \tag{4.32e}$$

$$(\Sigma_{k+1}^{-1})^p = (\Sigma_k^{-1})^p - (\Sigma_k^{-1^T})^p (\omega_k^p)^T \left(I + w_k^p (\Sigma_k^{-1})^p (w_k^p)^T\right)^{-1} \omega_k^p (\Sigma_k^{-1})^p \tag{4.32f}$$

$$(\Sigma_0^{-1})^p = \Sigma^{-1} \tag{4.32g}$$

$$\hat{\theta}_{k+1}^p = \text{Proj}\{\hat{\theta}_k^p + (\Sigma_k^{-1})^p (\omega_k^T)^p \left(I + w_k^p (\Sigma_k^{-1})^p (w_k^T)^p\right)^{-1} (e_k^p - \hat{\eta}_k^p), \Theta\} \tag{4.32h}$$

$$\tilde{\theta}^p = \theta - \hat{\theta}_k^p, \ \ \hat{\theta}_0^p = \hat{\theta} \tag{4.32i}$$

$$u^p(\tau) \triangleq \kappa(\tau, x^p(\tau), \hat{\theta}^p(\tau)) \in \mathbb{U} \tag{4.32j}$$

$$x^p(\tau) \in \mathbb{X}, \ \ x^p(T) \in \mathbb{X}_f(\tilde{\theta}^p(T)) \tag{4.32k}$$

As before, the effect of future parameter adaptation is also accounted for in this formulation but the proposed discrete-time parameter estimation and set-based approach is considered.

The conservativeness of the algorithm is reduced by parameterizing both W and $\mathbb{X}_f$ as functions of $\tilde{\theta}(T)$. While it is possible for the set Θ to contract upon θ over time, the robustness feature due to $\vartheta \in \mathcal{D}$ will still remain.

Algorithm 2 *The MPC algorithm performs as follows: At sampling instant k*

1. ***Measure*** *the current state of the plant x_k and obtain the current value of matrices w and Σ^{-1} from equations 4.2 and 4.10 respectively*

2. ***Obtain*** *the current value of parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from* (4.11) *and* (4.24) *respectively*

   If $\quad z_{\theta_k} \leq z_{\theta_{k-1}} - \left\|\hat{\theta}_k - \hat{\theta}_{k-1}\right\|$

$$\hat{\theta} = \hat{\theta}_k, \quad z_\theta = z_{\theta_k}$$

   Else

$$\hat{\theta} = \hat{\theta}_{k-1}, \quad z_\theta = z_{\theta_{k-1}}$$

   End

3. ***Solve*** *the optimization problem* (4.31) *and apply the resulting feedback control law to the plant until the next sampling instant*

4. ***Increment*** $k = k + 1$*. Repeat the procedure from step 1 for the next sampling instant.*

The min-max approach guarantees robust stability, but it remains impractical in practice due to the high computational burden. The next approach adopts a Lipschitz-based constraint, which retains robust stability at the cost of some conservatism.

4.4.2 Lipschitz-based Approach

In this section, we present a Lipschitz-based method whereby the nominal model rather than the unknown bounded system state is controlled subject to conditions that ensure that given constraints are satisfied for all possible uncertainties. State prediction error bound is determined based on the Lipschitz continuity of the model. A knowledge of appropriate Lipschitz bounds for the x-dependence of the dynamics $F(x_k, u_k)$ and $G(x_k, u_k)$ are assumed as follows:

Assumption 5 *A set of functions $\mathcal{L}_j : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^+$, $j \in \{F, G\}$ are known which satisfy*

$$\mathcal{L}_j(\mathbb{X}, u) \quad \geq \quad \min\left\{\mathcal{L}_j \;\middle|\; \sup_{x_1, x_2 \in \mathbb{X}} \Big(\|j(x_1, u) - j(x_2, u)\| - \mathcal{L}_j \|x_1 - x_2\| \Big) \quad \leq \quad 0\right\},$$

where for $j \equiv g$ is interpreted as an induced norm since $g(x, u)$ is a matrix.

Assuming a knowledge of the Lipschitz bounds for the x-dependence of the dynamics $F(x_k, u_k)$ and $G(x_k, u_k)$ as given in Assumption 5 and let $\Pi = z_\theta + \left\|\hat{\theta}\right\|$, a worst-case deviation $z^p_{x,k} \geq \max_{\theta\in\Theta} \|x_k - x^p_k\|$ can be generated from

$$z^p_{x,k+1} = (\mathcal{L}_f + \mathcal{L}_g\Pi)z^p_{x,k} + \|G(x^p_k, u_k)\| \, z_\theta + M_\vartheta, \;\; z^p_{x,0} = 0. \tag{4.34}$$

Using this error bound, the robust Lipschitz-based MPC is given by

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) = u^*(0) \tag{4.35a}$$

$$u^*(.) \triangleq \arg \min_{u^p_{[0,T]}} J(x, \hat{\theta}, z_\theta, u^p) \tag{4.35b}$$

where

$$\begin{aligned}
J(x, \hat{\theta}, z_\theta, u^p) \;\; &= \sum_{k=0}^{T-1} L(x^p_k, u^p_k)d\tau + W(x^p_T, z^p_\theta) && (4.36a)\\
s.t. \;\; \forall k &\in [0, \, T] \\
x^p_{k+1} &= x^p_k + F(x^p_k, u^p_k) + G(x^p_k, u^p_k)\hat{\theta}, \quad x^p_0 = x && (4.36b)\\
z^p_{x,k+1} &= (\mathcal{L}_f + \mathcal{L}_g\Pi)z^p_{x,k} + \|G^p(x^p_k, u^p_k)\| \, z_\theta + M_\vartheta, \; z^p_{x,0} = 0 && (4.36c)\\
X^p(\tau) &\triangleq B(x^p_k, z^p_{x,k}) \subseteq \mathbb{X}, \quad u^p_k \in \mathbb{U} && (4.36d)\\
X^p(T) &\subseteq \mathbb{X}_f(z_\theta) && (4.36e)
\end{aligned}$$

The effect of the disturbance is built into the uncertainty cone $B(x^p_k, z^p_{x,k})$ via (4.36c). Since the uncertainty bound is not monotonically decreasing in this case, the uncertainty radius z_θ which appears in (4.36c) and in the terminal expressions of (4.36a) and (4.36e) are held constant over the prediction horizon. However, the fact that they are updated at sampling instants when z_θ shrinks reduces the conservatism of the robust MPC and enlarges the terminal domain that would otherwise have been designed based on a large initial uncertainty z_{θ_0}.

Algorithm 3 *The Lipschitz-based MPC algorithm performs as follows: At sampling instant k*

1. ***Measure*** *the current state of the plant $x = x_k$*

2. ***Obtain*** *the current value of the parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from equations* (4.11) *and* (4.24) *respectively,*

   If $\quad z_{\theta_k} \leq z_{\theta_{k-1}}$

   $$\hat{\theta} = \hat{\theta}_k, \quad z_\theta = z_{\theta_k}$$

   Else

   $$\hat{\theta} = \hat{\theta}_{k-1}, \quad z_\theta = z_{\theta_{k-1}}$$

End

3. ***Solve*** *the optimization problem* (4.35) *and apply the resulting feedback control law to the plant until the next sampling instant*

4. ***Increment*** $k := k+1$*; repeat the procedure from step 1 for the next sampling instant.*

4.5 Close-loop Robust Stability

Robust stabilization to the target set Ξ is guaranteed by appropriate selection of the design parameters W and X_f. The robust stability conditions require the satisfaction of the following criteria.

Criterion 1 *The terminal penalty function* $W : \mathbb{X}_f \times \tilde{\Theta}^0 \rightarrow [0, +\infty]$ *and the terminal constraint function* $\mathbb{X}_f : \tilde{\Theta}^0 \rightarrow \mathbb{X}$ *are such that for each* $(\theta, \hat{\theta}, \tilde{\theta}) \in (\Theta^0 \times \Theta^0 \times \tilde{\Theta}^0_\epsilon)$*, there exists a feedback* $k_f(., \hat{\theta}) : \mathbb{X}_f \rightarrow \mathbb{U}$ *satisfying*

1. $0 \in \Xi \subseteq \mathbb{X}_f(\tilde{\theta}) \subseteq \mathbb{X}$, $\mathbb{X}_f(\tilde{\theta})$ *closed*

2. $k_f(x, \hat{\theta}) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f(\tilde{\theta})$

3. $W(x, \tilde{\theta})$ *is continuous with respect to* $x \in \mathbb{R}^{n_x}$

4. $\forall x \in \mathbb{X}_f(\tilde{\theta}) \backslash \Xi$, $\mathbb{X}_f(\tilde{\theta})$ *is strongly positively invariant under* $k_f(x, \hat{\theta})$ *with respect to* $x_+ \in x + F(x, k_f(x, \hat{\theta})) + G(x, k_f(x, \hat{\theta}))\Theta + \mathcal{D}$

5. $L(x, k_f(x, \hat{\theta})) + W(x_+, \hat{\theta}) - W(x, \hat{\theta}) \leq 0$, $\forall x \in \mathbb{X}_f(\tilde{\theta}) \backslash \Xi$.

The condition 5 from criteria 1 require W to be a local robust CLF for the uncertain system 4.1 with respect to $\theta \in \Theta$ and $\vartheta \in \mathcal{D}$.

Criterion 2 *For any* $\tilde{\theta}_1,\ \tilde{\theta}_2 \in \tilde{\Theta}^0$ *s.t.* $\left\|\tilde{\theta}_2\right\| \leq \left\|\tilde{\theta}_1\right\|$,

1. $W(x, \tilde{\theta}_2) \leq W(x, \tilde{\theta}_1),\ \ \forall x \in \mathbb{X}_f(\tilde{\theta}_1)$

2. $\mathbb{X}_f(\tilde{\theta}_2) \supseteq \mathbb{X}_f(\tilde{\theta}_1)$

4.5.1 Main Results

Theorem 2 *Let* $X_{d0} \triangleq X_{d0}(\Theta^0) \subseteq \mathbb{X}$ *denote the set of initial states with uncertainty* Θ^0 *for which* (4.31) *has a solution. Assuming criteria* 3 *and* 4 *are satisfied, then the closed-loop system state* x*, given by (4.1,4.3,4.2, 4.7,4.10, 4.11,4.24,4.31), originating from any* $x_0 \in X_{d0}$ *feasibly approaches the target set* Ξ *as* $t \rightarrow +\infty$.

Proof: *Feasibility*: The closed-loop stability is based upon the feasibility of the control action at each sample time. Assuming, at time t, that an optimal solution $u^p_{[0,T]}$ to the optimization problem (4.31) exist and is found. Let Θ^p denote the estimated uncertainty set at time t and Θ^v denote the set at time $t+1$ that would result with the feedback implementation of $u_t = u^p_0$. Also, let x^p represents the worst case state trajectory originating from $x^p_0 = x_t$ and x^v represents the trajectory originating from $x^v_0 = x + v$ for $v \in \Big\{F(x^a, u^p) + G(x^a, u^p)\Theta^b + \mathcal{D}\Big\}$ under the same feasible control input $u^v_{[1,T]} = u^p_{[1,T]}$. Moreover, let $X^a_{\Theta^b} \triangleq \{x^a \,|\, x^a_+ \in x^a + F(x^a, u^p) + G(x^a, u^p)\Theta^b + \mathcal{D}\}$ which represents the set of all trajectories of the uncertain dynamics.

Since the $u^p_{[0,T]}$ is optimal with respect to the worst case uncertainty scenario, it suffice to say that $u^p_{[0,T]}$ drives any trajectory $x^p \in X^p_{\Theta^p}$ into the terminal region $\mathbb{X}^p_f$. Since Θ is non-expanding over time, we have $\Theta^v \subseteq \Theta^p$ implying $x^v \in X^p_{\Theta^v} \subseteq X^p_{\Theta^p}$. The terminal region $\mathbb{X}^p_f$ is strongly positively invariant for the nonlinear system (4.1) under the feedback $k_f(.,.)$, the input constraint is satisfied in $\mathbb{X}^p_f$ and $\mathbb{X}^v_f \supseteq \mathbb{X}^p_f$ by criteria 1(2.), 1(4.) and 4(2.) respectively. Hence, the input $u = [u^p_{[1,T]}, k_{f[T,T+1]}]$ is a feasible solution of (4.31) at time $t+1$ and by induction, the optimization problem is feasible for all $t \geq 0$.

Stability: The stability of the closed-loop system is established by proving strict decrease of the optimal cost $J^*(x, \hat{\theta}, z_\theta) \triangleq J(x, \hat{\theta}, z_\theta, \kappa^*)$. Let the trajectories ($x^p$, $\hat{\theta}^p$, $\tilde{\theta}^p$, z^p_θ) and control u^p correspond to any worst case minimizing solution of $J^*(x, \hat{\theta}, z_\theta)$. If $x^p_{[0,T]}$ were extended to $k \in [0, T+1]$ by implementing the feedback $u^p_{T+1} = k_f(x^p_{T+1}, \hat{\theta}^p)$ then criterion 1(5) guarantees the inequality

$$L(x^p_T, k_f(x^p_T, \hat{\theta}^p_T)) + W(x^p_{T+1}, \tilde{\theta}^p_T) - W(x^p_T, \tilde{\theta}^p_T) \leq 0. \tag{4.37}$$

The optimal cost

$$\begin{aligned} J^*(x_t, \hat{\theta}_t, z_{\theta_t}) &= \sum_{k=0}^{T-1} L(x^p_k, u^p_k) + W(x^p_T, \tilde{\theta}^p_T) \geq \sum_{k=0}^{T-1} L(x^p, u^p) + W(x^p_T, \tilde{\theta}^p_T) \\ &\quad + L(x^p_T, k_f(x^p_T, \hat{\theta}^p_T)) + W(x^p_{T+1}, \tilde{\theta}^p_T) - W(x^p_T, \tilde{\theta}^p_T) \qquad (4.38) \\ &\geq L(x^p_0, u^p_0) + \sum_{k=1}^{T} L(x^p_k, u^p_k) \\ &\qquad + L(x^p_T, k_f(x^p_T, \hat{\theta}^p_T)) + W(x^p_{T+1}, \tilde{\theta}^p_{T+1}) \qquad (4.39) \\ &\geq L(x^p_0, u^p_0) + J^*(x_{t+1}, \hat{\theta}_{t+1}, z_{\theta_{t+1}}) \qquad (4.40) \end{aligned}$$

Then, it follows from (4.40) that

$$J^*(x_{t+1}, \hat{\theta}_{t+1}, z_{\theta_{t+1}}) - J^*(x_t, \hat{\theta}_t, z_{\theta_t}) \leq -L(x_t, u_t) \leq -\mu_L(\|x\|). \tag{4.41}$$

where μ_L is a class $\mathcal{K}_\infty$ function. Hence $x(t)$ enters Ξ asymptotically.

Remark 2 *In the above proof,*

- (4.38) *is obtained using inequality* (4.37)
- (4.39) *follows from criterion 1.1 and the fact that* $\left\|\tilde{\theta}\right\|$ *is non-increasing*
- (4.40) *follows by noting that the last 3 terms in* (4.39) *is a (potentially) suboptimal cost on the interval* $[\delta, T+\delta]$ *starting from the point* $(x^p(\delta), \hat{\theta}^p(\delta))$ *with associated uncertainty set* $B(\hat{\theta}^p(\delta), z_\theta^p(\delta))$.

The closed-loop stability is established by the feasibility of the control action at each sample time and the strict decrease of the optimal cost J^*. The proof follows from the fact that the control law is optimal with respect to the worst case uncertainty $(\theta,\ \vartheta) \in (\Theta,\ \mathcal{D})$ scenario and the terminal region $\mathbb{X}_f^p$ is strongly positively invariant for (4.1) under the (local) feedback $k_f(.,.)$. ■

Theorem 3 *Let* $X'_{d0} \triangleq X'_{d0}(\Theta^0) \subseteq \mathbb{X}$ *denote the set of initial states for which* (4.35) *has a solution. Assuming Assumption* 14 *and Criteria* 3 *and* 4 *are satisfied, then the origin of the closed-loop system given by (4.1,4.3,4.2, 4.7,4.10, 4.11,4.24,4.35) is feasibly asymptotically stabilized from any* $x_0 \in X'_{d0}$ *to the target set* Ξ.

The proof of the Lipschitz-based control law follows from that of theorem 2.

Remark 3 *Note that the min-max approach can be prohibitively difficult to implement in practice due to the computational complexity associated with min-max optimization algorithms. In contrast, the Lipschitz-based approach can be implemented using any standard real-time optimization algorithm currently used for the solution of standard MPC problems. This latter technique will be employed in the simulation example presented in the next section.*

Remark 4 *Since the additional Lipschitz constraints can be implemented using the standard NLP solvers, the computational burden of the Lipschitz-based approach is the same of usual constrained nonlinear MPC. Details about the scale-up and computational aspects of this problem can be found in GRUNE and PANNEK (2011).*

4.6 Simulation Results

4.6.1 CSTR with the van de Vusse kinetics

The van de Vusse reaction is a well known chemical system used to test nonlinear controllers because of the presence of undesirable dynamic behaviour, which includes gain inversion and non-minimum phase response. The reaction mechanism is described by:

$$\begin{aligned} &A \xrightarrow{k1} B \xrightarrow{k2} C \\ &2A \xrightarrow{k3} D \end{aligned} \tag{4.42}$$

In discrete-time, the nonisothermal CSTR dynamics can be represented by the equations 4.43. The system states are the concentrations of the components A (x_1) e B (x_2) and the reactor temperature (x_3). In addition, two manipulated variables were considered, the dilution rate (u_1) and the jacket temperature (u_2).

$$\begin{aligned} x_1(k+1) &= x_1(k) + \Delta t u_1(k)[C_{ae} - x_1(k)] - \Delta t \theta_1 e^{-\alpha_1/x_3(k)} x_1(k) \\ &\quad - \Delta t \theta_3 e^{-\alpha_3/x_3(k)} x_1^2(k) \\ x_2(k+1) &= x_2(k) - \Delta t u_1(k) x_2(k) + \Delta t \theta_1 e^{-\alpha_1/x_3(k)} x_1(k) \\ &\quad - \Delta t \theta_2 e^{-\alpha_2/x_3(k)} x_2(k) \\ x_3(k+1) &= x_3(k) + \frac{\Delta t}{\rho C_p} \Big(\theta_1 e^{-\alpha_1/x_3(k)} x_1(k) \Delta H_1 \\ &\quad + \theta_2 e^{-\alpha_2/x_3(k)} x_2(k) \Delta H_2 + \theta_3 e^{-\alpha_3/x_3(k)} x_1^2(k) \Delta H_3 \Big) \\ &\quad + \Delta t u_1(k)[T_0 - x_3(k)] + \Delta t \frac{K_w A_R}{\rho c_p V} [u_2(k) - x_3(k)] \end{aligned} \tag{4.43}$$

This system can be represented by:

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \mathbf{F}(\mathbf{x}(k), \mathbf{u}(k)) + \mathbf{G}(\mathbf{x}(k), \mathbf{u}(k)) \boldsymbol{\theta} + \mathbf{v}_k \tag{4.44}$$

For $\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]^T$, these matrices are defined as:

$$\mathbf{F}(\mathbf{x}(k), \mathbf{u}(k)) = \begin{pmatrix} u_1(k)[C_{ae} - x_1(k)] \\ -u_1(k) x_2(k) \\ u_1(k)[T_0 - x_3(k)] + \frac{K_w A_R}{\rho c_p V}[u_2(k) - x_3(k)] \end{pmatrix} \Delta t \tag{4.45}$$

$$\mathbf{G}(\mathbf{x}(k),\mathbf{u}(k)) = \begin{pmatrix} -e^{-\alpha_1/x_3(k)}x_1(k) & 0 & -e^{-\alpha_3/x_3(k)}x_1^2(k) \\ e^{-\alpha_1/x_3(k)}x_1(k) & -e^{-\alpha_2/x_3(k)}x_2(k) & 0 \\ \frac{e^{-\alpha_1/x_3(k)}x_1(k)\Delta H_1}{\rho C_p} & \frac{e^{-\alpha_2/x_3(k)}x_2(k)\Delta H_2}{\rho C_p} & \frac{e^{-\alpha_3/x_3(k)}x_1^2(k)\Delta H_3}{\rho C_p} \end{pmatrix}\Delta t \tag{4.46}$$

The fixed parameters were obtained from KLATT and ENGELL (1998) and are showed in Table 4.1. The (unknown) true value of the parameter vector is $\boldsymbol{\theta}_r = [1.287 \quad 1.287 \quad 9.043]^T$. The control objective is to regulate the

Table 4.1: Model parameters

Parameter	Value
α_1	9758.3 K
α_2	9758.3 K
α_3	8560.0 K
ΔH_1	4.2 $\frac{kJ}{molA}$
ΔH_2	-11 $\frac{kJ}{molB}$
ΔH_3	-41.85 $\frac{kJ}{molA}$
ρ	0.9342 kg/l
C_p	3.01 $\frac{kJ}{kg \cdot K}$
A_R	0.215 m^2
k_w	4032 $\frac{kJ}{hm^2K}$
T_0	403.15 K

desired product concentration (x_2) and the reactor temperature (x_3) to a setpoint and, simultaneously, estimate the frequency (or pre-exponential) factors of the Arrhenius equation, assumed to lie inside a ball of known radius. This is a real industrial problem found in reactors where catalyst deactivation is substantial and, consequently, the kinetics parameters may change after the system start-up. An example of this chemical system is the deactivation of hydro-treating catalysts by coke deposition PACHECO *et al.* (2011).

The MPC quadratic cost function can be written in deviation variables as:

$$\ell(\tilde{\mathbf{x}},\tilde{\mathbf{u}}) = \tilde{\mathbf{x}}^T\mathbf{Q}\tilde{\mathbf{x}} + \tilde{\mathbf{u}}^T\mathbf{R}\tilde{\mathbf{u}} \tag{4.47}$$

In which $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{eq}$ and $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_{eq}$. The subscript eq denotes the equilibrium point. The terminal penalty was a quadratic parameter dependent function:

$$W(\tilde{\mathbf{x}},\boldsymbol{\theta}) = \tilde{\mathbf{x}}^T\mathbf{P}(\boldsymbol{\theta})\tilde{\mathbf{x}} \tag{4.48}$$

It was obtained solving a finite set of linear matrix inequalities (LMI) as proposed in GAHINET *et al.* (1996). For this purpose, the MATLAB LMI toolbox MATLAB

(2008) was used to represent the system and to find the solution.

The terminal region was estimated by the algorithm presented in CHEN and ALLGÖWER (1998), which the main step is to find the value α that satisfies $\tilde{\mathbf{x}}^T\mathbf{P}(\boldsymbol{\theta})\tilde{\mathbf{x}} \leq \alpha$ by solving an optimization problem decreasing the value of α until the optimal value is nonpositive.

For the initial nominal estimate, the matrix $\mathbf{P}(\boldsymbol{\theta})$ is given by:

$$\mathbf{P}(\boldsymbol{\theta}) = \begin{bmatrix} 2.9776 & 0 & 0 \\ 0 & 2.9760 & 0 \\ 0 & 0 & 3.5916 \end{bmatrix} \tag{4.49}$$

Using this matrix, the terminal region optimization problem was solved. The maximum point was $\tilde{\mathbf{x}}_{max} = [2.9776 \;\; 2.9760 \;\; 3.5916]$. This solutions leads to the terminal region:

$$\tilde{\mathbf{x}}^T\mathbf{P}(\boldsymbol{\theta})\tilde{\mathbf{x}} \leq 177 \tag{4.50}$$

Open-loop tests of the parameter estimation routine

The uncertainty based estimation routine for discrete-time systems was tested for the estimation of the frequency factors in an open-loop test. Two scenarios were evaluated. In the first one, the disturbance added to the system is a pulse in the manipulated variables. In the second test, a persistent bounded periodical signal was added to the reactor temperature. In both cases, it is showed that the true parameters values were recovered. The initial parameter vector estimate was:

$$\boldsymbol{\theta}_0 = [5 \;\; 6 \;\; 7]^T \tag{4.51}$$

In the first simulation, the disturbance inserted into the system is a 10% pulse in the jacket temperature as showed in figure 4.1. In Figures 4.2, 4.3 and 4.4, the

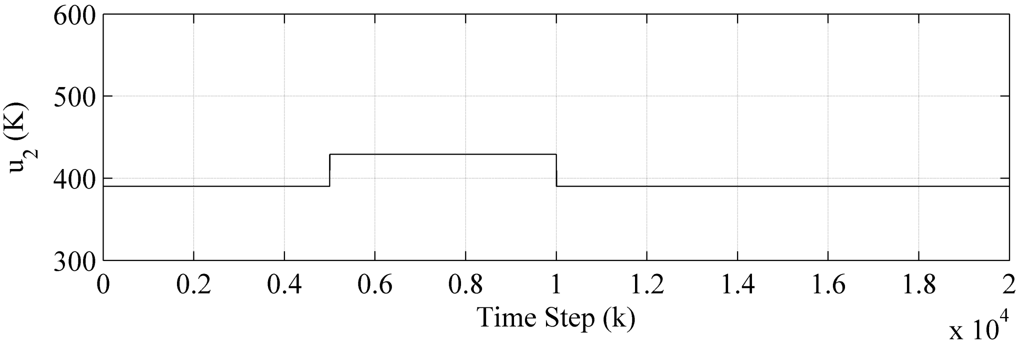

Figure 4.1: Disturbance inserted in the jacket temperature u_2

time evolution of the parameters during the simulation is showed. The excitation added by the pulse disturbance improves the convergence and accelerates the set contraction as showed in figure 4.6. The true values of the parameters are recovered and the prediction error converges to zero (Figure 4.5)

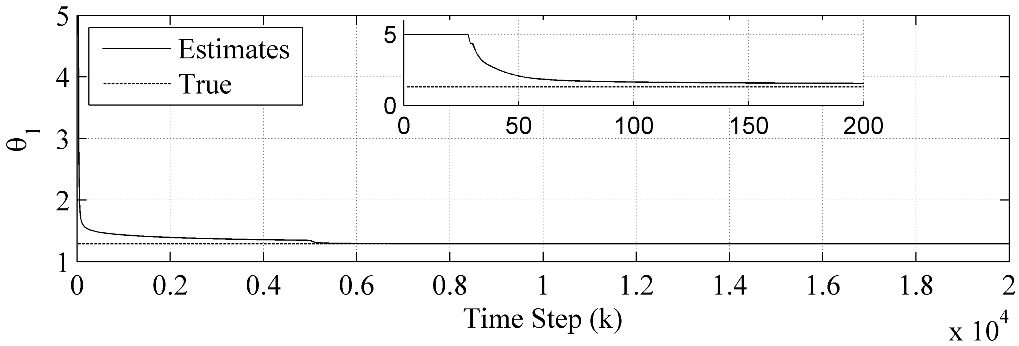

Figure 4.2: Time evolution of the parameter estimates and true values, using a pulse as disturbance ($\boldsymbol{\theta}_1$)

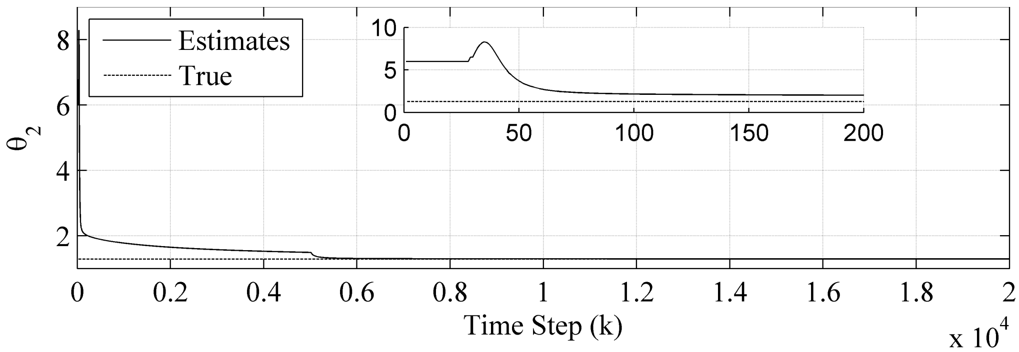

Figure 4.3: Time evolution of the parameter estimates and true values, using a pulse as disturbance ($\boldsymbol{\theta}_2$)

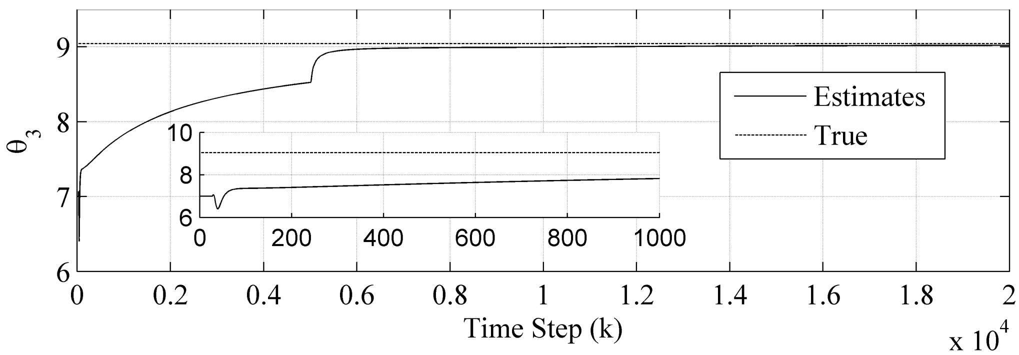

Figure 4.4: Time evolution of the parameter estimates and true values, using a pulse as disturbance ($\boldsymbol{\theta}_3$)

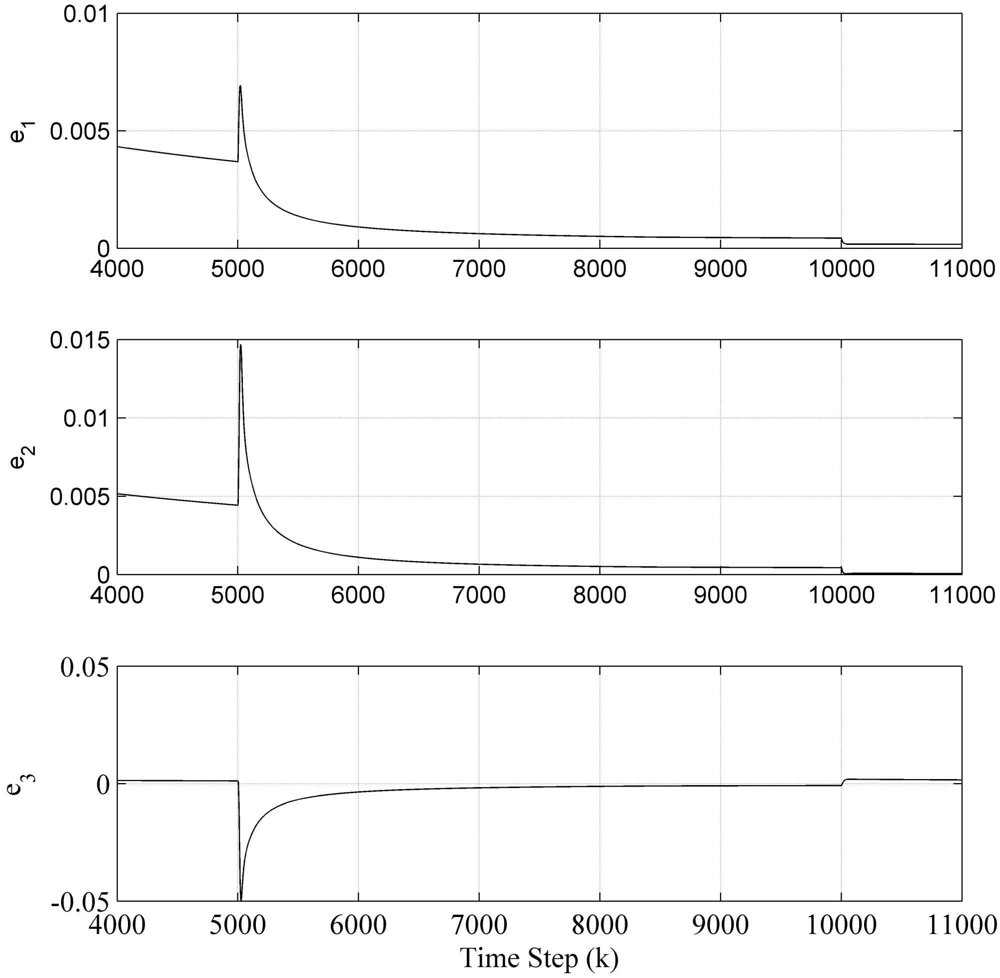

Figure 4.5: State prediction error $e_k = x_k - \hat{x}_k$ versus time step (k)

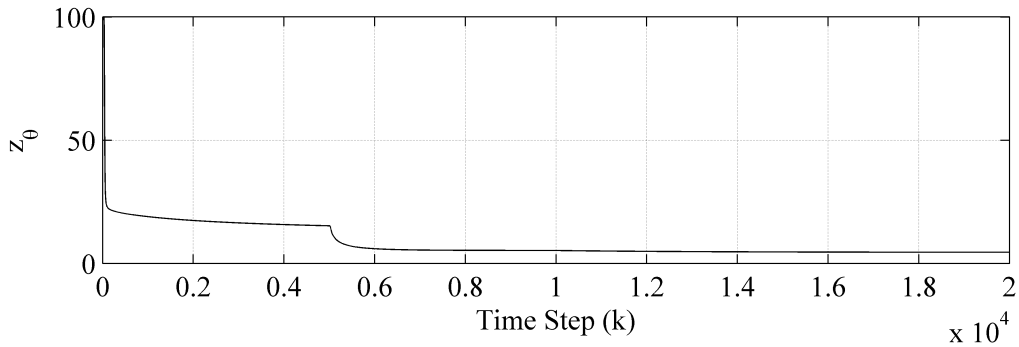

Figure 4.6: Progression of the set radius

In the following simulation, a persistent periodical disturbance was added to the jacket temperature:

$$u_2(k) = u_{2,nom} + B \cdot \sin\left(\frac{k}{C}\right) \tag{4.52}$$

The parameters were $B = 1$ and $C = 50$; the time course of the jacket temperature is showed in Figure 4.7.

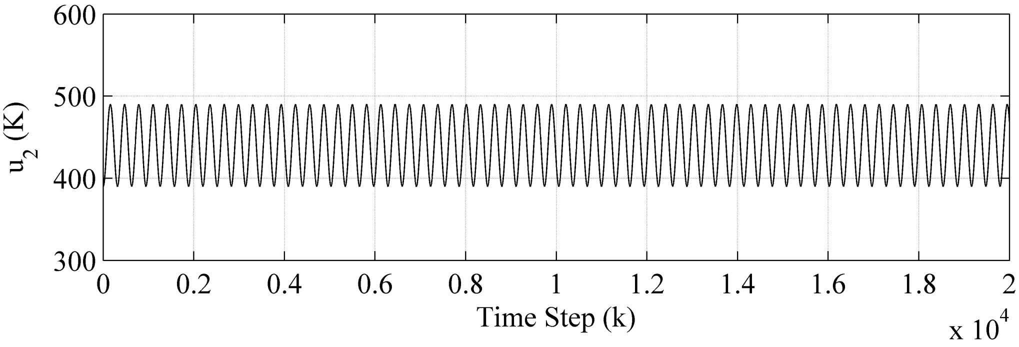

Figure 4.7: Sine disturbance inserted in the jacket temperature u_2

The sine disturbance provides more excitation to the reactor system than the pulse signal. Accordingly, it leads to a faster convergence to the true values. The parameters convergence are showed in the Figures 4.8, 4.9 and 4.10. Due to the difference in the rates of parameters convergence, the beginning of the simulation is showed in the subfigure. As one can see, the kinetic constant of the side product reaction ($2A \xrightarrow{k3} D$), represented by the parameter θ_3 (Figure 4.10), has the slowest rate of convergence. Finally, Figure 4.12 shows the estimation error converging to the origin. Figure 4.11 displays the decreasing of the uncertainty set radius versus time-step.

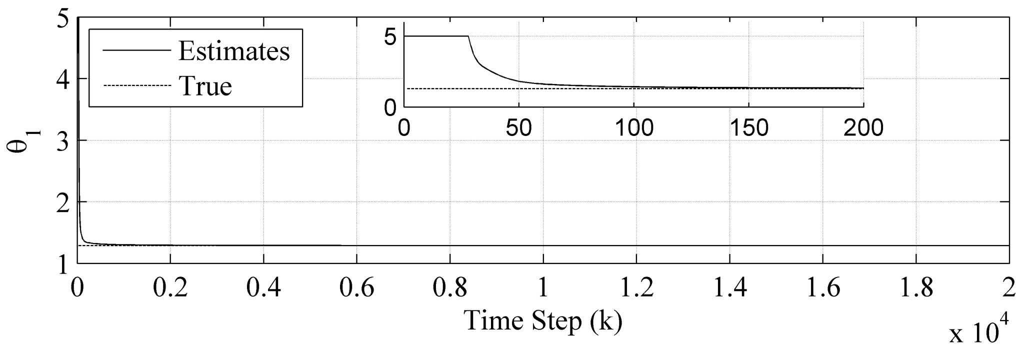

Figure 4.8: Time evolution of the parameter estimates and true values, using a sine function as disturbance ($\boldsymbol{\theta}_1$)

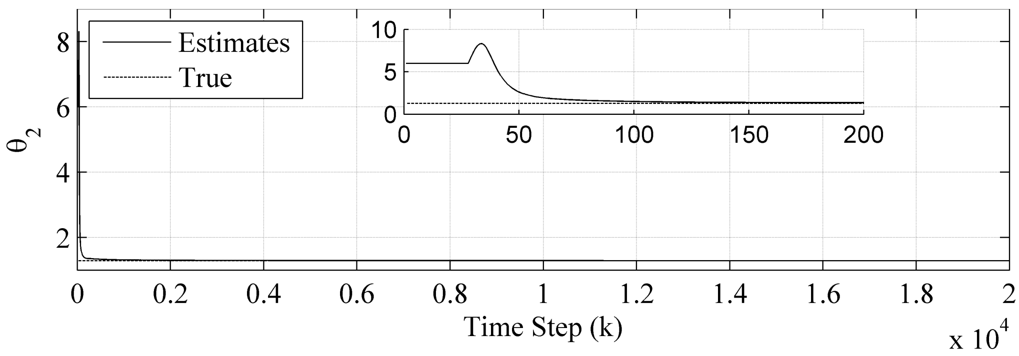

Figure 4.9: Time evolution of the parameter estimates and true values, using a sine function as disturbance ($\boldsymbol{\theta}_2$)

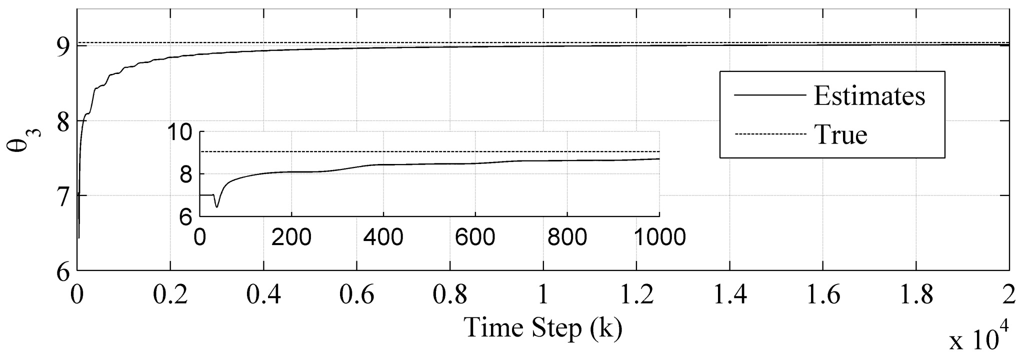

Figure 4.10: Time evolution of the parameter estimates and true values, using a sine function as disturbance ($\boldsymbol{\theta}_3$)

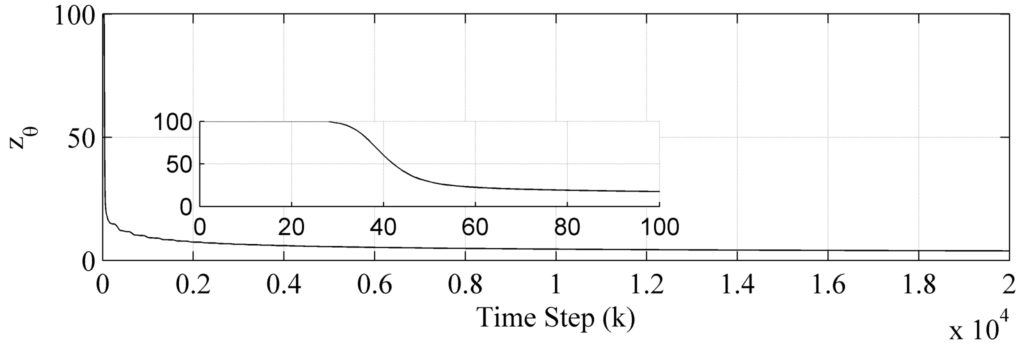

Figure 4.11: Progression of the set radius using a sine function as disturbance

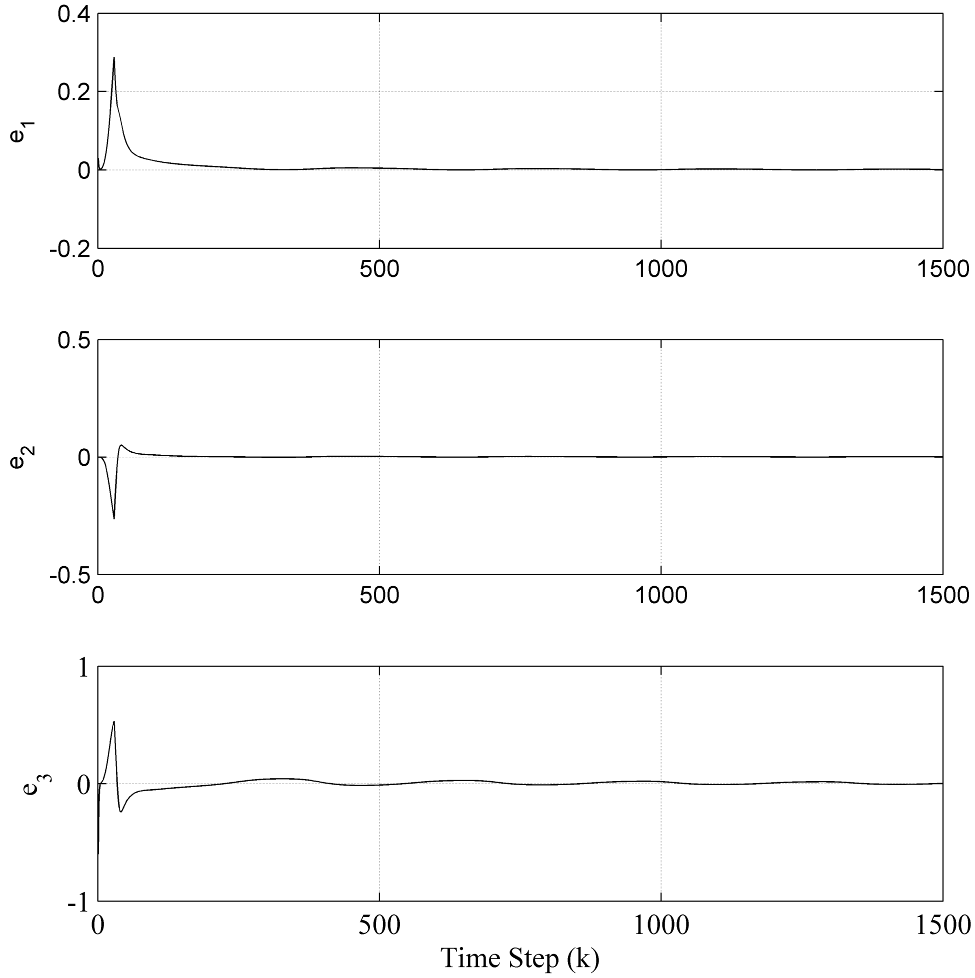

Figure 4.12: State prediction error $e_k = x_k - \hat{x}_k$ versus time step (k) for a sine disturbance

Closed-loop simulations

For the closed-loop simulations, the Lipschitz constraint was used to design a robust control system. The initial values of the parameters are assumed to lie in a ball of radius $z_{\hat{\boldsymbol{\theta}}0} = 25$ centred at the initial estimate $\boldsymbol{\theta}_0 = [5 \;\; 6 \;\; 7]^T$. The true value of the parameter vector is $\boldsymbol{\theta}_r = [1.287 \;\;\; 1.287 \;\;\; 9.043]^T$. The matrices for the cost function 4.47 were chosen to be:

$$\mathbf{Q} = \begin{bmatrix} 35 & 0 \\ 0 & 7 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.53}$$

The control objective is to regulate the system to the setpoint:

$$\mathbf{x}_{sp} = [x_{2,sp} \quad x_{3,sp}] = [1.079(mol/l) \quad 394.7(K)] \tag{4.54}$$

while satisfying the constraints:

$$\begin{aligned} 0 \leq u_1 \leq 500 \\ 0 \leq u_2 \leq 600 \\ 0 \leq \Delta u_1 \leq 10 \\ 0 \leq \Delta u_2 \leq 30 \\ 0 \leq x_2 \leq 10 \\ 0 \leq x_3 \leq 800 \end{aligned} \tag{4.55}$$

In Figures 4.13 and 4.14 the concentration and reactor temperature are showed. The state variables achieve the desired setpoint without offset. The manipulated variables are showed in Figures 4.15 (reactor flowrate) and 4.16 (jacket temperature). Moreover, the parameter estimates converge to the true values, the parameter θ_3 has the slowest convergence rate (Figures 4.18, 4.19 and 4.20). Figure 4.17 shows the uncertainty set radius reduction over time.

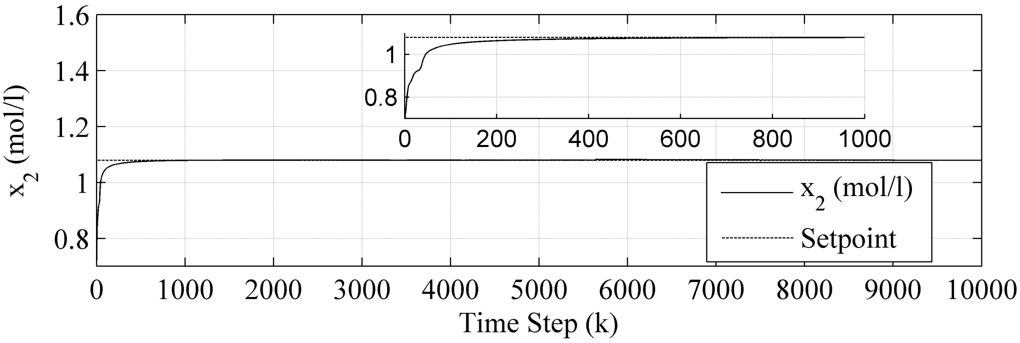

Figure 4.13: Concentration trajectory for the closed-loop system (x_2) versus time

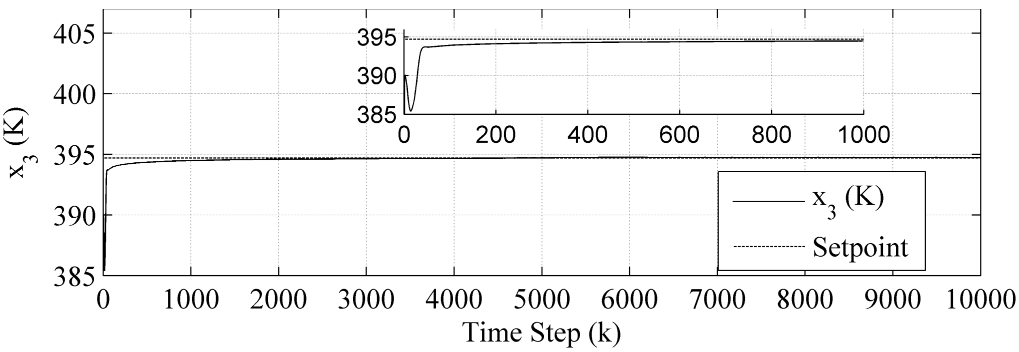

Figure 4.14: Reactor temperature trajectory for the closed-loop system (x_3) versus time

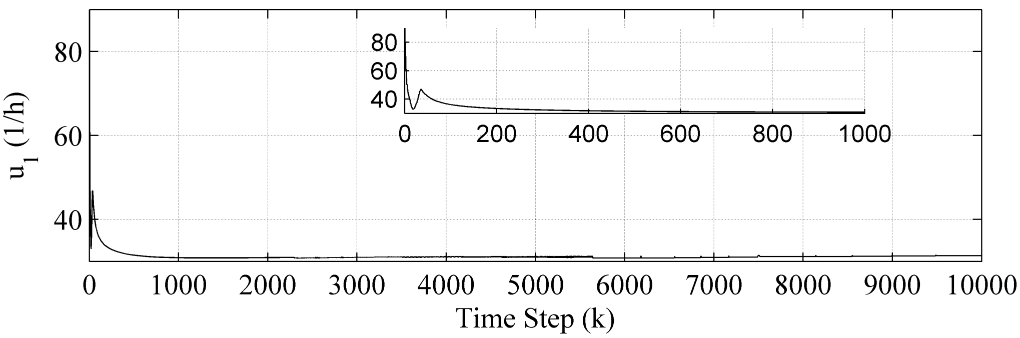

Figure 4.15: Manipulated flowrate (u_1) versus time

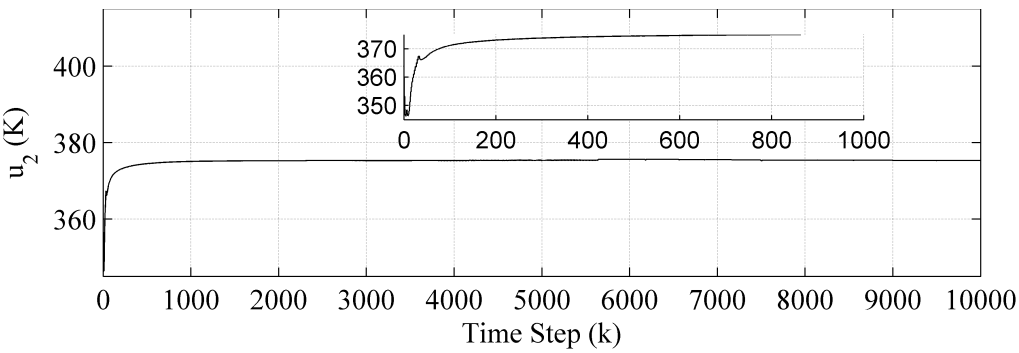

Figure 4.16: Manipulated jacket temperature (u_2) versus time

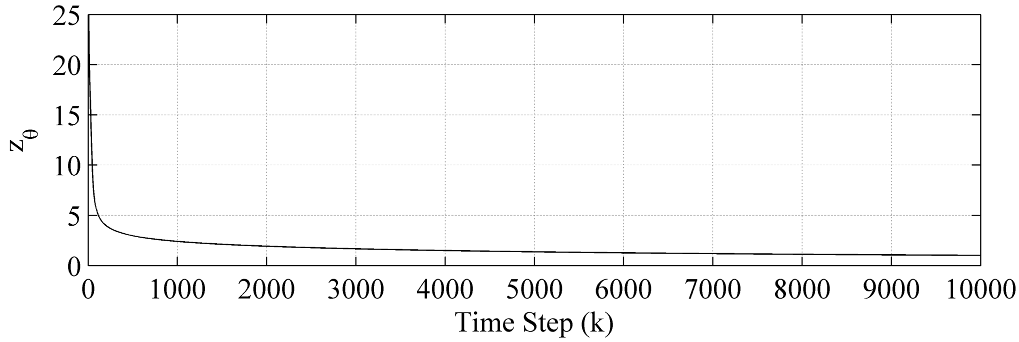

Figure 4.17: Closed-loop set radius versus time

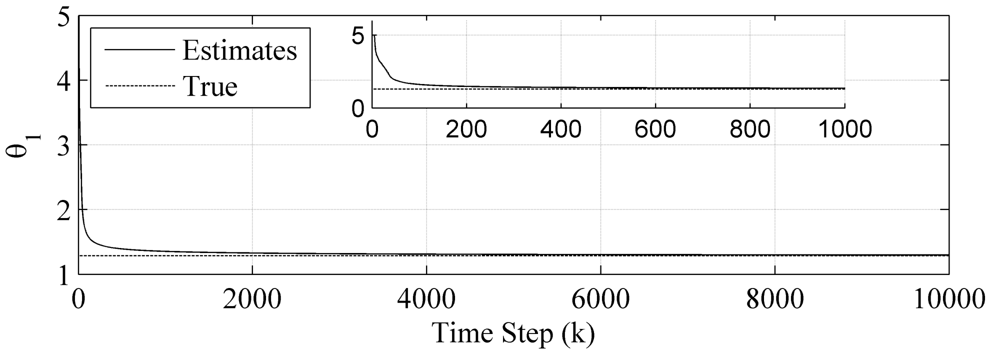

Figure 4.18: Time evolution of the parameter estimates and true values for the closed-loop system ($\boldsymbol{\theta}_1$)

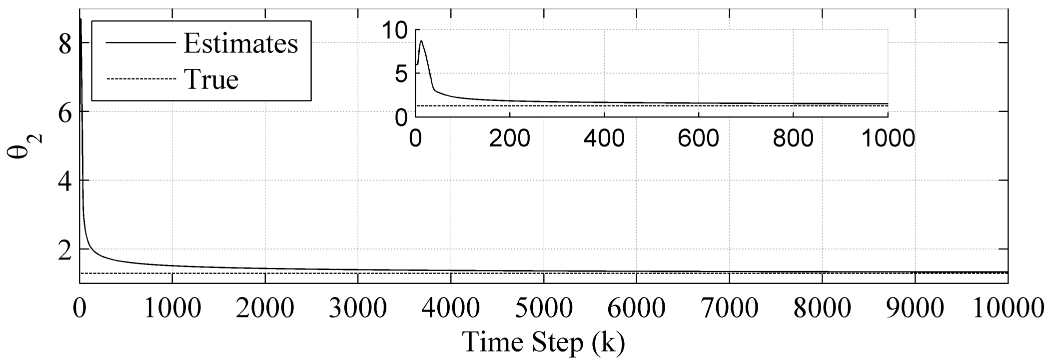

Figure 4.19: Time evolution of the parameter estimates and true values for the closed-loop system ($\boldsymbol{\theta}_2$)

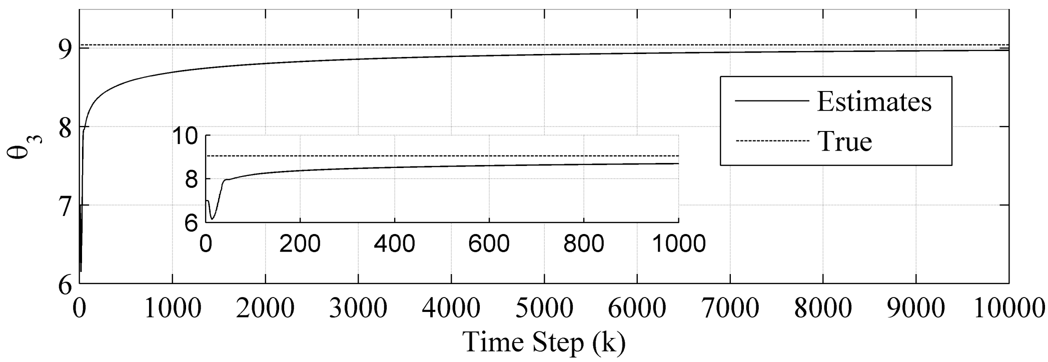

Figure 4.20: Time evolution of the parameter estimates and true values for the closed-loop system ($\boldsymbol{\theta}_3$)

Closed-loop simulations with disturbances

In order to simulate a disturbance in the closed-loop system, a fluctuation in the inlet temperature was introduced as a periodic function:

$$T_0(k) = 403.15 + sin(k) \tag{4.56}$$

The control objective is to regulate the system to the set-point presented in section 4.6.1. As depicted in Figures 4.26, 4.27, 4.28, and 4.25, the parameter convergence and the set radius reduction is more conservative in comparison with the disturbance free case (Figures 4.18, 4.19, 4.20 and 4.17). The true values of the parameters, however, are recovered. As expected, the reactor temperature and concentration oscillate around the set-point (Figures 4.22 and 4.21). In Figures 4.23 and 4.24, the control actions are showed.

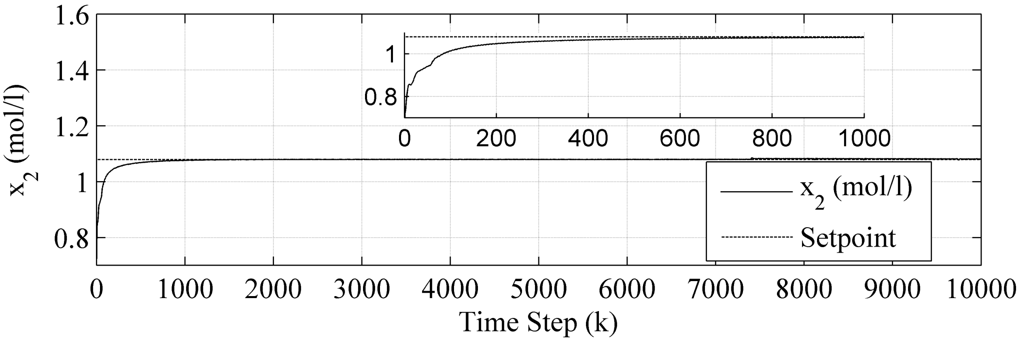

Figure 4.21: Concentration trajectory for the closed-loop system (x_2) with disturbance versus time

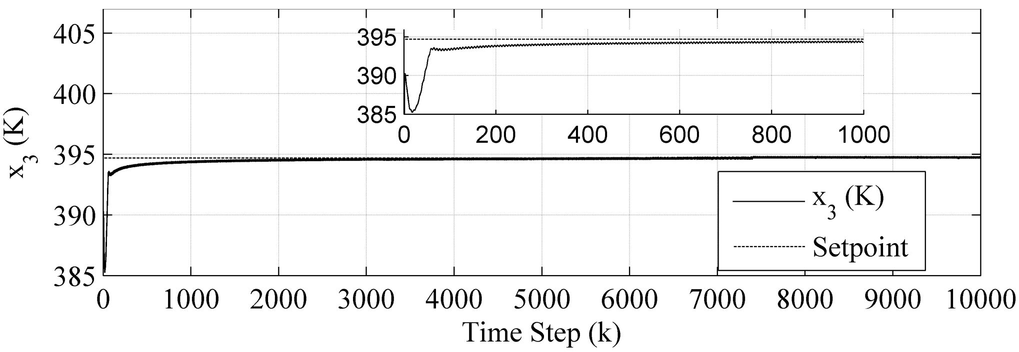

Figure 4.22: Reactor temperature trajectory for the closed-loop system (x_3) with disturbance versus time

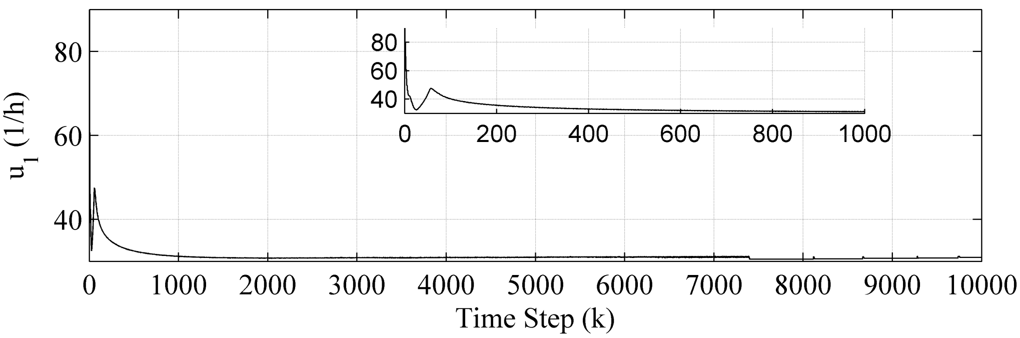

Figure 4.23: Manipulated flowrate (u_1) with disturbance versus time

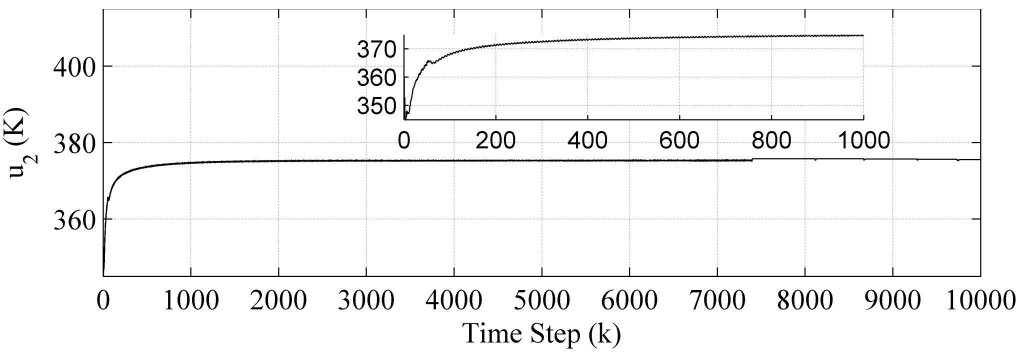

Figure 4.24: Manipulated jacket temperature (u_2) with disturbance versus time

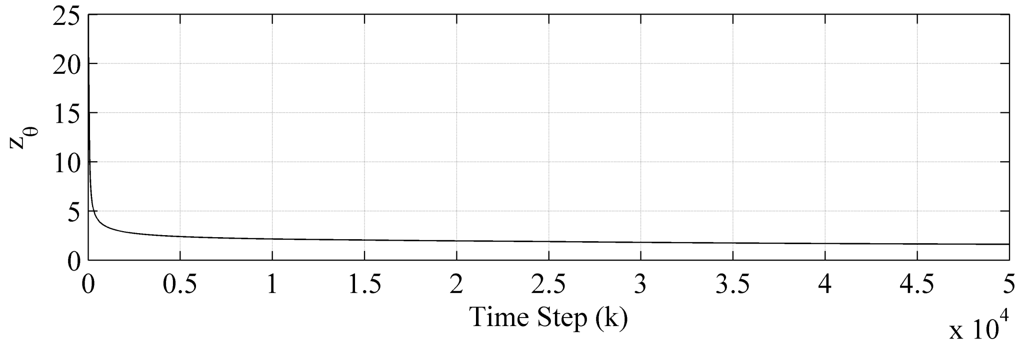

Figure 4.25: Closed-loop set radius versus time

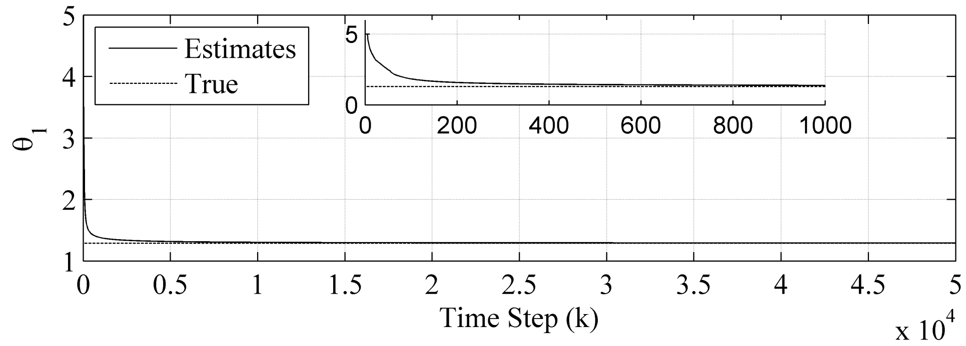

Figure 4.26: Time evolution of the parameter estimates and true values for the closed-loop system ($\boldsymbol{\theta}_1$)

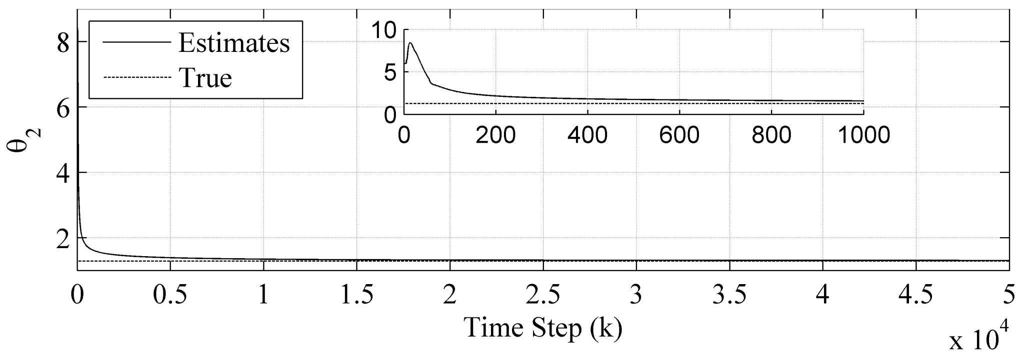

Figure 4.27: Time evolution of the parameter estimates and true values for the closed-loop system ($\boldsymbol{\theta}_2$)

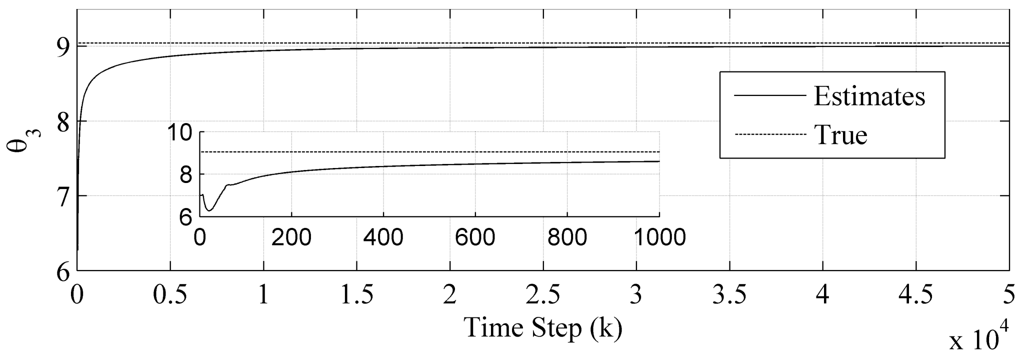

Figure 4.28: Time evolution of the parameter estimates and true values for the closed-loop system ($\boldsymbol{\theta}_3$)

4.6.2 Chemotherapy Control

The dynamics of a tumor model in a chemotherapy cancer treatment can be described by the following set of differential equations UPRETI (2012):

$$
\begin{aligned}
\frac{dy_1}{dt} &= u(t) - \gamma_6 y_1 \\
\frac{dy_2}{dt} &= \dot{y}_{2,in} + r_2 \frac{y_2 y_4}{\beta_2 + y_4} - \gamma_3 y_2 y_4 - \gamma_4 y_2 - \alpha_2 y_2 (1 - e^{-y_1 \lambda_2}) \\
\frac{dy_3}{dt} &= r_3 y_3 (1 - \beta_3 y_3) - \gamma_5 y_3 y_4 - \alpha_3 y_3 (1 - e^{-y_1 \lambda_3}) \\
\frac{dy_4}{dt} &= r_1 y_4 (1 - \beta_1 y_4) - \gamma_1 y_3 y_4 - \gamma_2 y_2 y_4 - \alpha_1 y_4 (1 - e^{-y_1 \lambda_1})
\end{aligned}
\tag{4.57}
$$

where y_1 is the drug concentration; y_2 the number of immune cells; y_3 the number of normal cells and y_4 the number of cancer cells. The variables in this model are in rescaled units (details can be viewed in PILLIS and RADUNSKAYA (2003)). The model includes immune response, since the immune cells can grow in the presence of tumour cells. It also includes competition terms for immune and cancer cells in the form of a predator-prey approach. In this formulation, the drug is able to kill all types of cells at different rates. A set of parameters and details about the interesting dynamic behaviour of this model is presented in PILLIS and RADUNSKAYA (2003). Table 4.2 reproduces a set of parameters for a normal case.

Following the same approach of the previous example, the continuous equations were discretized to fit into the proposed class of discrete-time nonlinear models (Equation 4.1), the sampling time was set to 0.1 days. In this problem, the control objective is to minimize the tumour cells (y_4) during a treatment period while keeping the normal cells above a minimum value. A secondary goal is to drive the drug amount (y_1) to a minimum after the tumour is diminished (represented by the use of a small weight in the matrix $\mathbf{Q}$ for y_1 in comparison with y_4). For the

Table 4.2: Model parameters for the chemotherapy model from PILLIS and RADUNSKAYA (2003)

Parameter	Value
α_1	0.3
α_2	0.2
α_3	0.1
γ_1	1
γ_2	0.5
γ_3	1
γ_4	0.2
γ_5	1
γ_6	1
r_1	1.5
r_2	0.01
r_3	1
β_1	1
β_2	0.3
β_3	1
$\dot{y}_{2,in}$	0.33

adaptive MPC framework, two parameters were chosen to be unknown: γ_3 and γ_5. The objective function can be stated as:

$$\ell(\bar{\mathbf{y}}, \bar{\mathbf{u}}) = \bar{\mathbf{y}}^T \mathbf{Q} \bar{\mathbf{y}} + \Delta \mathbf{u}^T \mathbf{R} \Delta \mathbf{u} + W(\bar{\mathbf{y}}, \boldsymbol{\theta}) \tag{4.58}$$

with $\bar{y} = [y_1 \quad y_2 \quad y_3 \quad y_4]$ and the matrices:

$$\mathbf{Q} = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \mathbf{R} = 10 \tag{4.59}$$

The minimum normal cells constraint was added to the problem formulation in the Lipschitz framework (Equations 4.36):

$$\mathbf{y}_3 > 0.75 \tag{4.60}$$

The terminal cost $W(\bar{\mathbf{y}}, \boldsymbol{\theta})$ and the terminal region were obtained using the approach proposed in the previous example, using the model linearization and solving a set of LMI in a LPV framework GAHINET *et al.* (1996). Following PILLIS and RADUNSKAYA (2003), the initial condition is a patient with cancer and in the basis of attraction of a large tumour burden. These condition imply the eventual death of the patient in the absence of drug treatment. Figure 4.29 shows the model simulation using the initial condition $y(0) = [0 \quad 0.1 \quad 1.5 \quad 0.25]$ and no drug. The same initial condition was used in the closed-loop simulations.

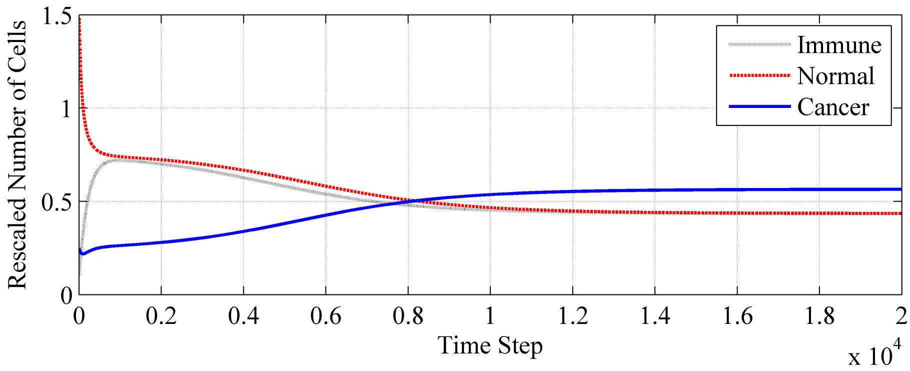

Figure 4.29: Open-loop simulation of the chemotherapy model for a initial condition in the basis of attraction of a tumor growth.

Two sets of simulations were performed, in the first one no disturbance was added to the system. For the second case, a sinusoidal disturbance in the form of equation 4.61 (A = 0.01 and m = 1) was added to the tumour measurement.

$$\vartheta_4(k) = A\sin(mk) \tag{4.61}$$

The initial estimate of the parameter vector was chosen as:

$$\begin{bmatrix}\gamma_3(0) & \gamma_5(0)\end{bmatrix}^T = [0.75 \quad 1.1]^T \tag{4.62}$$

Figure 4.30 shows the cancer cells behaviour using the drug dose obtained by the use of the adaptive NMPC algorithm (Figure 4.31). As one can notice, the algorithm provides a drug dose that is able to drive the tumour cells to a neighbourhood around the origin. The drug dose is reduced automatically by the algorithm after the tumour decreases. The disturbed case has a overshoot in the drug dose in comparison with the disturbance free case. As pictured in Figure 4.32, in both cases the minimum healthy cells constraint is satisfied. Finally, the parameter estimation results are showed in Figures 4.33 and 4.34. The true values of the parameters are recovered in both cases, moreover, the uncertainty set decreases along time with a final value slightly lower for the disturbance free scenario.

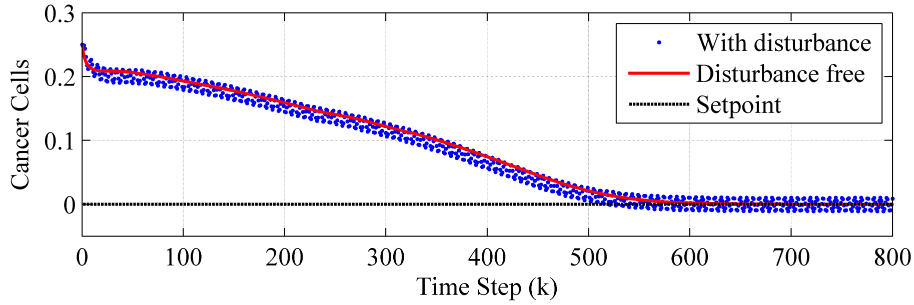

Figure 4.30: Time plot of the rescaled number of cancer cells using the predictive controller for drug dose calculation

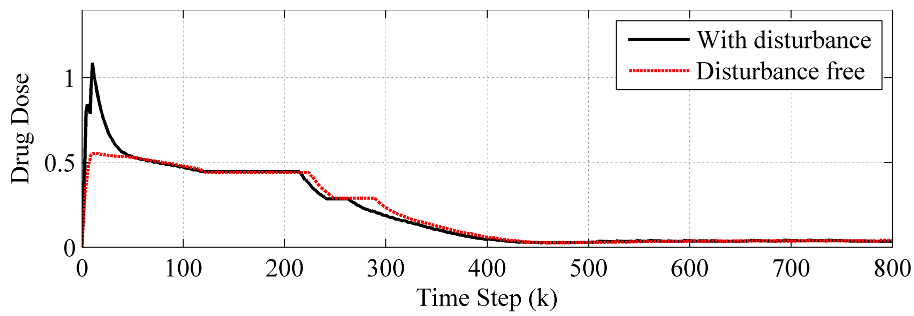

Figure 4.31: Drug doses along time using the predictive controller

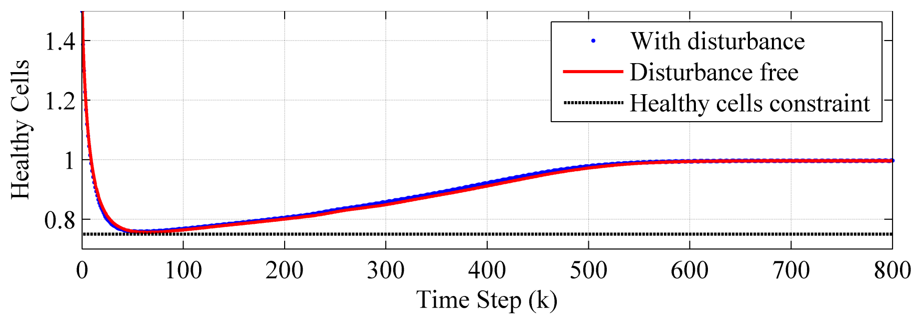

Figure 4.32: Healthy cells along time and the halthy cells constraint for the closed-loop simulations

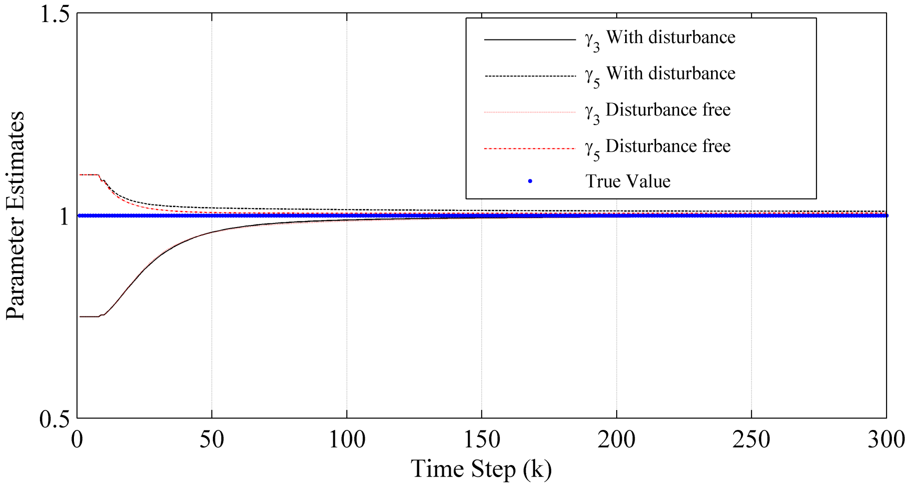

Figure 4.33: Parameter estimates along time for the closed-loop system.

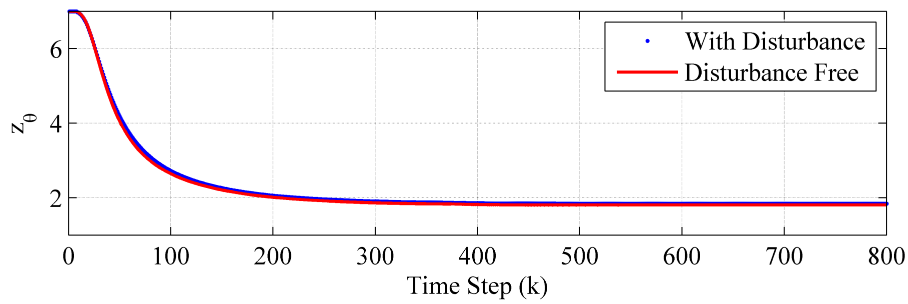

Figure 4.34: Uncertainty radius along time for the closed-loop system

4.7 Final Remarks

An adaptive NMPC design technique is proposed for the control of constrained discrete-time nonlinear systems subject to both parametric and time varying disturbances. The proposed robust controller updates the plant model online when model improvement is guaranteed. The adaptation mechanism enables the construction of terminal design parameters based upon subsets of the original parametric uncertainty in a minimally conservative approach. The conservativeness and the complexity due to the parametric uncertainty is effectively reduced over time using a self-exciting mechanism arising from the adaptive NMPC formulation. The portion due to the disturbance $\vartheta \in \mathcal{D}$ remains active for all time with guaranteed robust stability. Finally, the simulation results for the uncertainty set estimation and robust control show a good performance (with guaranteed stability) for two challenging nonlinear control problems with initially unknown parameters.

Chapter 5

A set-based estimation and robust adaptive output feedback model predictive control

5.1 Introduction

Despite the successful applications of model predictive control (MPC), especially in the process control industry, the theoretical development of this strategy is mainly based on full state measurement assumption. However, in many engineering applications, the mathematical description of process dynamics leads to dynamical system models that require a large number of state variables and unknown parameters. In general, full state measurements are either physically impossible or very costly. Furthermore, the models used in MPC approaches are not, in general, updated on-line and any change in the model structure can render the controller obsolete. One attractive solution for this problem is to use the available plant measurements to estimate the parameters and unknown states of the system in an adaptive output feedback strategy. Moreover, if an uncertainty bound is estimated on-line, a robust control problem, uncertainty based, can be formulated and solved in an MPC framework.

Once the values of the state variables are required for the reliable performance of a control system, they must be estimated using process measurements. The presence of unknown parameters further complicates the situation, especially if the performance of the control system relies heavily on the knowledge of the unknown parameters. Consequently, the parameters must also be estimated using available process information. Once the parameters and states are obtained, in order to use a robust control system, an uncertainty description is very useful. However, the uncertainty propagation processes are (in general) computational intensive, precluding the real-time application in a control strategy. Moreover, constrained control problems based on models (e.g. MPC) need state predictions in a time-horizon that may not be easy to compute in a uncertain situation, leading to a scenario that constraints can be violated.

A dynamical system whose state variables are estimates of the state variables of another dynamical system is called an observer FRIEDLAND (2005). State estimation is a broad and well established field of study. The Kalman filter KALMAN (1960) and its many variants JULIER *et al.* (2000) have been widely applied for large classes of linear and nonlinear systems. In the presence of unknown parameters, adaptive observers have been formulated to achieve joint estimation of parameters and state variables KUDVA and NARENDRA (1973). The main concept exploited in such techniques is to use the action of the observer by extending the state of the systems to treat constant (or slowly time-varying) parameters as additional state variables. This strategy has been used in concert with the Luenberger observer and Kalman filter. In recent work, an interval class of observers was proposed for systems with uncertain parameters (MAZENC and BERNARD (2011), MAZENC and DINH (2014)). The objective of this class of observers is to provide estimates of dynamic bounds for the state estimates that reflect the uncertain in the process parameters. This class was initially proposed in GOUZE *et al.* (2000). This technique has many applications, for instance, robust control and system biology.

The set estimation is suitable for the solution of robust control problems that rely on a law based on uncertainty description for the control action computation such as robust adaptive model predictive control design techniques. In this strategy, the state uncertainty can be used to guarantee robust constraint satisfaction SUBRAMANIAN *et al.* (2015) or to solve a worst-case control problem. Although, the estimation of uncertainty is a broad field of research, most existing techniques are concerned with detailed descriptions of the uncertainty set, which leads to a high computational burden for real-time applications DAHLIWAL and GUAY (2014).

Despite the fact that the state variables are not accessible for direct measurement in most industrial problems, a large part of existing nonlinear MPC theory is focused on the assumption of full state measurement and, in some cases, perfect measurement. A more useful scenario is to consider the problem in which discrete-time measurements are available for some of the state variables. In this case, a state estimation technique can be used in a nominal strategy where the state is replaced by the estimated state MAYNE (2014). In the context of continuous-time systems, IMSLAND *et al.* (2003) propose the use of high-gain observers to achieve output feedback stabilization of a class of nonlinear systems. In ADETOLA and GUAY (2003) a modified version of the high-gain observer was used to control sampled data systems. The robust constraint satisfaction problem was addressed by

SUBRAMANIAN *et al.* (2015), in this work a scenario-based MPC is proposed to solve the robust problem. Furthermore, a Taylor model technique is used to over-approximate the reachable set of the differential equations used for state prediction by the MPC controller.

A set-based state estimation algorithm for continuous-time systems and constant parameters was recently proposed in DHALIWAL and GUAY (2014). This study proposes an adaptive observer design that incorporates a set-based parameter estimation routine first proposed by ADETOLA *et al.* (2009). In the present work, an extension of the set-based adaptive observer approach is generalized to deal with a class of discrete-time nonlinear systems with constant and time-varying parameters. The proposed algorithm provides a simultaneous estimation of state variables and parameters along with an estimation of their uncertain sets. Under the assumption that the initial estimates of these uncertainty sets contain the values of the unknown parameter and state variables, convergence of the state and parameter estimates to the true values is guaranteed. The uncertainty set algorithm guarantees that the forward invariance of the unknown true value of the states and parameters is preserved through the contraction of the uncertainty sets. Moreover, this deterministic worst-case uncertainty allows the algorithm to be used in real-time applications. By combination of the developed estimation technique and a Lipschitz-based method, an over-approximation for the reachable sets of the difference equations can be computed. Finally, these sets are used in the MPC framework to guarantee constraint satisfaction and, furthermore, dynamic improvement for the conservatism, since the uncertainty sets are guaranteed to contract with time. Stability is obtained by considering the state estimation error as disturbance that can be computed directly using the set-based estimation. This chapter is organized as follows. The set-based state and parameter estimation are described in Section 5.2.1. In Section 5.3, the output feedback approach for MPC using the set-based estimation is showed. The closed-loop robust stability is proposed in Section 5.4, followed by a simulation study. Finally, the conclusions are presented in Section 5.5.

5.2 State and Parameter Estimation [1]

In the next two subsections the approaches for joint estimation of state and parameters are presented. In Subsection 5.2.1, the case for constant parameters is presented, subsequently, in Subsection 5.2.2 the time-varying case is showed.

[1]This section was partially presented at the American Control conference 2016 and was published in its annals (GONÇALVES *et al.*, 2016)

5.2.1 Constant Parameters

Problem Statement

In this section, we consider the following class of nonlinear discrete-time systems:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{x}(k) + \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{y}(k))\boldsymbol{\theta} + \mathbf{w}(k) \\ \mathbf{y}(k) &= \mathbf{H}\mathbf{x}(k) + \mathbf{v}(k)\end{aligned} \tag{5.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^y$ and $\boldsymbol{\theta} \in \mathbb{R}^p$. Additionally, the pair $(\mathbf{A}, \mathbf{H})$ is observable. As one can notice, the main characteristic of this structure is the nonlinear term depending only on the output. This discrete-time form is directly related with the continuous-time form presented in ATASSI and KHALIL (1999). This structure can arise from the transformation of the class:

$$\begin{aligned}\mathbf{x}(k+1) &= f(\mathbf{x}(k)) + g(\mathbf{x}(k))\theta \\ \mathbf{y}(k) &= h(\mathbf{x}(k))\end{aligned} \tag{5.2}$$

Conditions for the existence of transformations of nonlinear systems of the form (5.2) to the simplified structure (5.1) have been established using differential geometric techniques as presented in ISIDORI (1995).

Assumption 6 *The disturbances $\mathbf{w}(k)$ e $\mathbf{v}(k)$ are bounded. Moreover, the upper bound is given by the known constant $0 < M_v < \infty$.*

Assumption 7 *The parameters are uniquely identifiable[2]. At the initial time-step, they are contained in the compact set $\Theta(0) = B(\boldsymbol{\theta}(0), z_\theta(0))$, where $\boldsymbol{\theta}(0)$ is the initial parameter estimate and $z_{\boldsymbol{\theta}}(0)$ the ball radius.*

Assumption 8 *The true state value is within the set defined by the ball $B(\hat{\mathbf{x}}(0), \frac{z_\eta}{2})$, where $\hat{\mathbf{x}}(0)$ is the initial estimate and $z_\eta(0)$ the set radius.*

Set-based estimation

Consider the following observer for the states of Equation 5.1:

$$\begin{aligned}\hat{\mathbf{x}}(k+1) = {} & \hat{\mathbf{x}}(k) + \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}(\mathbf{y}(k))\hat{\boldsymbol{\theta}}(k+1) + \mathbf{K}(\mathbf{y}(k) - \mathbf{H}\hat{\mathbf{x}}(k)) \\ & - \mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) - (\mathbf{A} - \mathbf{K}\mathbf{H})\mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1))\end{aligned} \tag{5.3}$$

The $\mathbf{C}(k) \in \mathbb{R}^{p\times n}$ dynamics is given by the difference equation:

[2] $(y(k, \boldsymbol{\theta}_1) = y(k, \boldsymbol{\theta}_2), \ \ \forall k \geq k_0 \rightarrow \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2)$

$$\mathbf{C}^T(k+1) = \mathbf{C}^T(k) + (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k) + \mathbf{B}(\mathbf{y}(k)), \tag{5.4}$$

with initial conditions $\mathbf{C}(0) = \mathbf{0}$. Using the parametric error definition $\tilde{\boldsymbol{\theta}}(k) = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(k)$, where $\hat{\boldsymbol{\theta}}(k) \in \mathbb{R}^p$ is the parameter estimate at time-step k, we can write the state estimation error:

$$\begin{aligned}\mathbf{e}(k+1) = \mathbf{e}(k) + (\mathbf{A} - \mathbf{KH})\mathbf{e}(k) + \mathbf{B}(\mathbf{y}(k))\tilde{\boldsymbol{\theta}}(k+1) + \mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) \\ + (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) \underbrace{-\mathbf{Kv}(k) + \mathbf{w}(k)}_{\bar{\mathbf{w}}(k)}\end{aligned} \tag{5.5}$$

Consider the following auxiliary variable definition:

$$\boldsymbol{\eta}(k+1) = \mathbf{e}(k+1) - \mathbf{C}^T(k+1)\tilde{\boldsymbol{\theta}}(k+1) \tag{5.6}$$

Using (5.4) and (5.5), the auxiliary variable dynamics is given by:

$$\boldsymbol{\eta}(k+1) = \boldsymbol{\eta}(k) + (\mathbf{A} - \mathbf{KH})\boldsymbol{\eta}(k) + \bar{\mathbf{w}}(k) \tag{5.7}$$

Since the disturbance $\bar{\mathbf{w}}$ is unknown, an estimate of the auxiliary variable is computed using the following:

$$\begin{aligned}\hat{\boldsymbol{\eta}}(k+1) &= \hat{\boldsymbol{\eta}}(k) + (\mathbf{A} - \mathbf{KH})\hat{\boldsymbol{\eta}}(k) \\ \hat{\boldsymbol{\eta}}(0) &:= \underbrace{\bar{\mathbf{x}}(0) - \hat{\mathbf{x}}(0) = \bar{\mathbf{e}}(0)}_{\text{worst case error}}\end{aligned} \tag{5.8}$$

Using the initial estimate $\hat{\mathbf{x}}(0)$, $\bar{\mathbf{x}}(0)$ can be taken by any value distant $\frac{z_\eta}{2}$. The auxiliary variable error dynamics, $\tilde{\boldsymbol{\eta}}(k) = \boldsymbol{\eta}(k) - \hat{\boldsymbol{\eta}}(k)$, is given by:

$$\tilde{\boldsymbol{\eta}}(k+1) = \tilde{\boldsymbol{\eta}}(k) + (\mathbf{A} - \mathbf{KH})\tilde{\boldsymbol{\eta}}(k) + \bar{\mathbf{w}}(k) \tag{5.9}$$

For the initial time-step ($k = 0$):

$$\begin{aligned}\tilde{\boldsymbol{\eta}}(0) &= \boldsymbol{\eta}(0) - \hat{\boldsymbol{\eta}}(0) \\ &= \mathbf{e}(0) - \bar{\mathbf{e}}(0) \\ &= (\mathbf{x}(0) - \hat{\mathbf{x}}(0)) - (\bar{\mathbf{x}}(0) - \hat{\mathbf{x}}(0))\end{aligned} \tag{5.10}$$

Using the auxiliary variable definition and the assumption 8, it can be showed that the auxiliary variable error is bounded:

$$\begin{aligned}||\tilde{\boldsymbol{\eta}}(0)|| &\leq ||(\mathbf{x}(0)-\hat{\mathbf{x}}(0))-(\bar{\mathbf{x}}(0)-\hat{\mathbf{x}}(0))|| \\ &\leq ||(\mathbf{x}(0)-\hat{\mathbf{x}}(0))||+||(\bar{\mathbf{x}}(0)-\hat{\mathbf{x}}(0))|| \\ &\leq z_{\boldsymbol{\eta}} \end{aligned} \tag{5.11}$$

Since in the initial time-step $\mathbf{C}(0)=0$, $\boldsymbol{\eta}(0)=\mathbf{e}(0)$, the initial bound in the auxiliary variables implies a bound for the state estimation error. A conservative worst-case uncertainty can be chosen satisfying the inequality $||(\bar{\mathbf{x}}(0)-\hat{\mathbf{x}}(0))|| \leq z_{\boldsymbol{\eta}}/2$.

Uncertainty set update

We can write the error in the auxiliary variable as a linear disturbed system:

$$\tilde{\boldsymbol{\eta}}(k+1)=(\mathbf{I}+\mathbf{A}-\mathbf{KH})\tilde{\boldsymbol{\eta}}(k)+\bar{\mathbf{w}}(k). \tag{5.12}$$

A compact form can be written as:

$$\tilde{\boldsymbol{\eta}}(k+1)=\tilde{\mathbf{A}}\tilde{\boldsymbol{\eta}}(k)+\bar{\mathbf{w}}(k) \tag{5.13}$$

where $\tilde{\mathbf{A}}=(\mathbf{I}+\mathbf{A}-\mathbf{KH})$. The gain matrix $\mathbf{K}$ is chosen such that $\tilde{\mathbf{A}}$ is Hurwitz.

Consider the following Lyapunov function candidate:

$$V(k)=\tilde{\boldsymbol{\eta}}(k)^T\mathbf{P}\tilde{\boldsymbol{\eta}}(k) \tag{5.14}$$

Using (5.12), the rate of the change of the Lyapunov function candidate is as follows:

$$\begin{aligned}V(k+1)-V(k)=&\tilde{\boldsymbol{\eta}}(k)^T\underbrace{(\tilde{\mathbf{A}}^T\mathbf{P}\tilde{\mathbf{A}}-\mathbf{P})}_{-\bar{\mathbf{Q}}}\tilde{\boldsymbol{\eta}}(k)+\tilde{\boldsymbol{\eta}}(k)^T\tilde{\mathbf{A}}^T\mathbf{P}\bar{\mathbf{w}}(k) \\ &+\bar{\mathbf{w}}^T(k)\mathbf{P}\tilde{\mathbf{A}}\tilde{\boldsymbol{\eta}}(k)+\bar{\mathbf{w}}^T(k)\mathbf{P}\bar{\mathbf{w}}\end{aligned} \tag{5.15}$$

As a result, one can write the following:

$$V(k+1)-V(k)=-\tilde{\boldsymbol{\eta}}(k)^T\bar{\mathbf{Q}}\tilde{\boldsymbol{\eta}}(k)+2\tilde{\boldsymbol{\eta}}(k)^T\tilde{\mathbf{A}}^T\mathbf{P}\bar{\mathbf{w}}(k)+\bar{\mathbf{w}}^T(k)\mathbf{P}\bar{\mathbf{w}}$$

Each term on the right hand side of the previous inequality can be upper bounded as follows. The first term can be written as:

$$\tilde{\boldsymbol{\eta}}(k)^T\bar{\mathbf{Q}}\tilde{\boldsymbol{\eta}}(k)\geq\lambda_{min}(\mathbf{Q})\tilde{\boldsymbol{\eta}}(k)^T\tilde{\boldsymbol{\eta}}(k)\geq\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}V(k), \tag{5.16}$$

the second term as:

$$\begin{aligned}\tilde{\boldsymbol{\eta}}(k)^T\tilde{\mathbf{A}}^T\mathbf{P}\bar{\mathbf{w}}(k) &\leq \frac{1}{2}\tilde{\boldsymbol{\eta}}(k)^T\tilde{\boldsymbol{\eta}}(k)+\frac{1}{2}\bar{\mathbf{w}}(k)^T\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}\bar{\mathbf{w}}(k)\\ &\leq \frac{1}{2\lambda_{min}(\mathbf{P})}V(k)+\frac{||\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}||}{2}M_v, \end{aligned} \tag{5.17}$$

and, finally, the third term becomes:

$$\bar{\mathbf{w}}^T\mathbf{P}\bar{\mathbf{w}} \leq \lambda_{max}(\mathbf{P})M_v. \tag{5.18}$$

Using the above last inequalities, the rate of change of the Lyapunov function is bounded as follows:

$$V(k+1)-V(k) \leq -\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}-\frac{1}{2\lambda_{min}(\mathbf{P})}\right)V(k)+(||\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}||+\lambda_{max}(\mathbf{P}))M_v. \tag{5.19}$$

In the following, we consider an uncertainty set update for the auxiliary variables, $\tilde{\eta}_k$. The update is based on the evaluation of the following quantity:

$$z_{\boldsymbol{\eta}}(k)=\sqrt{\frac{V_{z\eta}(k)}{\lambda_{min}(\mathbf{P})}} \tag{5.20}$$

where $V_{z\eta}(k)$ is obtained from the recursion:

$$V_{z\boldsymbol{\eta}}(k+1)=V_{z\boldsymbol{\eta}}(k)-\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}-\frac{1}{2\lambda_{min}(\mathbf{P})}\right)V_{z\boldsymbol{\eta}}(k)+(||\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}||+\lambda_{max}(\mathbf{P}))M_v \tag{5.21}$$

with initial condition $V_{z\boldsymbol{\eta}}(0)=\lambda_{max}(\mathbf{P})z_{\boldsymbol{\eta}}(0)^2$.

Using (5.20) and (5.21), Algorithm 4 is proposed to estimate an uncertainty set for $\tilde{\eta}_k$.

Algorithm 4 *Uncertainty set update* $\mathcal{X}(k) \triangleq B(0, z_{\boldsymbol{\eta}}(k))$

1. *Initialize* $z_{\boldsymbol{\eta}}(0)=z_{\boldsymbol{\eta}0}$

2. *At time-step* k*, update:*

$$\mathcal{X}(k+1)=\begin{cases} B\left(0, z_{\boldsymbol{\eta}}(k)\right), & \textit{if } z_{\boldsymbol{\eta}}(k) \leq z_{\boldsymbol{\eta}}(k-1)\\ B\left(0, z_{\boldsymbol{\eta}}(k-1)\right), & \textit{otherwise}\end{cases} \tag{5.22}$$

3. $k=k+1$*, iterate back to step 2*

The following Lemma proposes that (5.20), (5.21) and Algorithm 4 can be used to provide an uncertainty set update for $\tilde{\eta}_k$ that guarantees containment of the unknown quantity η_k and forward invariance.

Lemma 5 *The uncertainty set evolution $\mathcal{X}(k) \triangleq B(0, z_{\boldsymbol{\eta}}(k))$ satisfies:*

1. $\tilde{\eta}(0) \in \mathcal{X}(0) \rightarrow \tilde{\eta}(k) \in \mathcal{X}(k)$

2. $\mathcal{X}(k+1) \subseteq \mathcal{X}(k)$

Proof:

1. *By definition: $V(0) \leq V_{z\boldsymbol{\eta}}(0)$. We know that $\Delta V(k) \leq \Delta V_{z\boldsymbol{\eta}}(k)$, which yields:*

$$V(k) \leq V_{z\boldsymbol{\eta}}(k) \quad \forall k \geq 0. \tag{5.23}$$

Using the definition of $V(k)$, it follows that:

$$V(k) = \tilde{\boldsymbol{\eta}}(k)^T \mathbf{P} \tilde{\boldsymbol{\eta}}(k) \geq \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) \lambda_{min}(\mathbf{P})$$

From the last inequality, we obtain:

$$\frac{V(k)}{\lambda_{min}(\mathbf{P})} \geq \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) \tag{5.24}$$

such that $z_{\boldsymbol{\eta}}(k)$ yields:

$$z_{\boldsymbol{\eta}}^2(k) = \frac{V_{z\boldsymbol{\eta}}(k)}{\lambda_{min}(\mathbf{P})} \geq \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) \quad \forall k \geq 0 \tag{5.25}$$

It follows that if $\tilde{\boldsymbol{\eta}}(0) \in \mathcal{X}(0)$, then $\tilde{\boldsymbol{\eta}}(k) \in B(0, z_{\boldsymbol{\eta}}(k)) \quad \forall k \geq 0$, as required.

2. *If $\mathcal{X}(k+1) \nsubseteq \mathcal{X}(k)$ then:*

$$\sup_{\tilde{\boldsymbol{\eta}} \in \mathcal{X}(k+1)} ||\tilde{\boldsymbol{\eta}}(k)|| \geq \mathbf{z}_{\boldsymbol{\eta}}(k) \tag{5.26}$$

To obtain $\Delta V_{z\boldsymbol{\eta}}(k) \leq 0$, one must ensure that:

$$V_{z\boldsymbol{\eta}}(k) \geq \frac{(||\mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P}|| + \lambda_{max}(\mathbf{P}))M_v}{\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2\lambda_{min}(\mathbf{P})}\right)} \tag{5.27}$$

At time-step $k+1$, one has:

$$V_{z\boldsymbol{\eta}}(k+1) \leq \max \left[V_{z\boldsymbol{\eta}}(k), \frac{(||\mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P}|| + \lambda_{max}(\mathbf{P}))M_v}{\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2\lambda_{min}(\mathbf{P})}\right)} \right] \tag{5.28}$$

and, consequently, a bound of the form:

$$z_{\boldsymbol{\eta}}(k+1) \leq \max \left[z_{\boldsymbol{\eta}}(k), \sqrt{\frac{(||\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}|| + \lambda_{max}(\mathbf{P}))M_v}{\left(\frac{\lambda_{min}(\mathbf{Q})\lambda_{min}(\mathbf{P})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2}\right)}} \right]. \tag{5.29}$$

Since the algorithm only updates the set if a contraction, $z_{\boldsymbol{\eta}}(k+1) \leq z_{\boldsymbol{\eta}}(k)$, is achieved, we have:

$$\sup_{\tilde{\boldsymbol{\eta}} \in \mathcal{X}(k+1)} ||\tilde{\boldsymbol{\eta}}(k)|| \leq \mathbf{z}_{\boldsymbol{\eta}}(k+1) \leq \mathbf{z}_{\boldsymbol{\eta}}(k) \tag{5.30}$$

which is a contradiction of (5.26)*. The result of the lemma follows.*

■

Parameter Estimation

The parameter estimation routine considered in this study is closely related to the continuous-time technique initially developed by ADETOLA *et al.* (2009).

Parameter Update

Consider the following identification matrix $\Sigma(k) \in \mathbb{R}^{p\times p}$:

$$\Sigma(k+1) = \Sigma(k) + \omega^T(k)\omega(k), \quad \Sigma(0) = \alpha\mathbf{I} \succ 0 \tag{5.31}$$

$\omega(k) \in \mathbb{R}^{y\times p}$ is given by:

$$\omega(k) = \mathbf{H}\mathbf{C}^T(k). \tag{5.32}$$

Consider also the following error and auxiliary variable associated with outputs:

$$\begin{aligned} \mathbf{e}_y(k) &= \mathbf{H}\mathbf{e}(k), \\ \hat{\boldsymbol{\eta}}_y(k) &= \mathbf{H}\hat{\boldsymbol{\eta}}(k). \end{aligned} \tag{5.33}$$

Using the auxiliary variable definition, it is possible to write:

$$\begin{aligned} \mathbf{H}\boldsymbol{\eta}(k+1) &= \mathbf{H}\mathbf{e}(k+1) - \mathbf{H}\mathbf{C}^T(k+1)\tilde{\boldsymbol{\theta}}(k+1) \\ &= \mathbf{e}_y(k+1) - \omega(k+1)\tilde{\boldsymbol{\theta}}(k+1). \end{aligned} \tag{5.34}$$

The inverse of $\Sigma(k)$ can be computed recursively as follows:

$$\Sigma^{-1}(k+1) = \Sigma^{-1}(k) - \Sigma^{-1^T}(k)\omega^T(k) \underbrace{\left(\mathbf{I} + w(k)\Sigma^{-1}(k)w(k)^T\right)^{-1}}_{\mathbf{L}(k)} \omega(k)\Sigma^{-1}(k),$$

$$\Sigma^{-1}(0) = \frac{1}{\alpha}\mathbf{I} \succ 0. \tag{5.35}$$

Consider the following parameter adaptation law based on a implicit regression model GONÇALVES and GUAY (2016):

$$\begin{aligned} \hat{\boldsymbol{\theta}}(k+1) &= \hat{\boldsymbol{\theta}}(k) + \Sigma^{-1}(k)\omega^T(k)\mathbf{L}(k)(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k)) \\ \bar{\hat{\boldsymbol{\theta}}}(k+1) &\triangleq \text{Proj}\{\hat{\boldsymbol{\theta}}(k+1), \Theta(k)\}. \end{aligned} \tag{5.36}$$

The projection algorithm is designed following GOODWIN and SIN (1984) to ensure that:

- $\hat{\boldsymbol{\theta}}(k+1) \in \Theta(k)$
- $\bar{\tilde{\boldsymbol{\theta}}}(k+1)^T\Sigma(k+1)\bar{\tilde{\boldsymbol{\theta}}}(k+1) \leq \tilde{\boldsymbol{\theta}}(k+1)^T\Sigma(k+1)\tilde{\boldsymbol{\theta}}(k+1)$

The following lemma is required in the analysis of the convergence of the parameter estimation scheme.

Lemma 6 *HADDAD and VIJAYSEKHAR (2008) Consider the system*

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{5.37}$$

where $\mathbf{A}$ *is a stable matrix with eigenvalues inside the unit circle and* $\mathbf{B}$ *is a matrix of appropriate dimension. Then, it can be shown that*

$$\sum_{k=0}^{K-1} \mathbf{x}(k+1)^T\mathbf{x}(k+1) \leq \delta^2 \sum_{k=0}^{K-1} \mathbf{u}(k)^T\mathbf{u}(k) \tag{5.38}$$

for some $\delta > 0$ *and* $K - 1 > 0$.

Let l_2 denote the space of square finitely summable signals and consider the following lemma.

Lemma 7 *The identifier (5.31) and parameter update law (5.36) are such that* $\tilde{\boldsymbol{\theta}}(k) = \boldsymbol{\theta}(k) - \hat{\boldsymbol{\theta}}(k)$ *is bounded. Furthermore, if*

$$\bar{\mathbf{w}} \in l_2 \quad or \quad \sum_{k=0}^{\infty}[\left\|\tilde{\boldsymbol{\eta}}_y(k)\right\|^2 - \gamma\left\|\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k)\right\|^2] < +\infty \tag{5.39}$$

for some $\gamma > 0$ *and*

$$\lim_{k\to\infty} \Sigma(k) = \infty \tag{5.40}$$

are satisfied, then $\tilde{\boldsymbol{\theta}}(k)$ converges to $\mathbf{0}$ asymptotically.

Proof: Let $V_{\tilde{\theta}}(k) = \tilde{\boldsymbol{\theta}}(k)^T \Sigma(k) \tilde{\boldsymbol{\theta}}(k)$. It follows from the properties of the projection operator that:

$$\begin{aligned} V_{\tilde{\boldsymbol{\theta}}}(k+1) - V_{\tilde{\boldsymbol{\theta}}}(k) = & \bar{\tilde{\boldsymbol{\theta}}}^T(k+1)\Sigma(k+1)\bar{\tilde{\boldsymbol{\theta}}}(k+1) - \tilde{\boldsymbol{\theta}}^T(k)\Sigma(k)\tilde{\boldsymbol{\theta}}(k) \\ & \leq \tilde{\boldsymbol{\theta}}^T(k+1)\Sigma(k+1)\tilde{\boldsymbol{\theta}}(k+1) - \tilde{\boldsymbol{\theta}}^T(k)\Sigma(k)\tilde{\boldsymbol{\theta}}(k). \end{aligned} \tag{5.41}$$

Using the parameter update law and the Equation 5.34, one can write $\tilde{\boldsymbol{\theta}}(k+1)$ as:

$$\tilde{\boldsymbol{\theta}}(k+1) = \tilde{\boldsymbol{\theta}}(k) - \Sigma^{-1}(k)\omega^T(k)\mathbf{L}(k)(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k))$$

Upon substitution of $e_y(k) = \eta(k) + \omega(k)\tilde{\boldsymbol{\theta}}(k)$, one obtains:

$$= \tilde{\boldsymbol{\theta}}(k) - \Sigma^{-1}(k)\omega^T(k)\mathbf{L}(k)(\omega(k)\tilde{\boldsymbol{\theta}}(k) + \tilde{\boldsymbol{\eta}}_y(k))$$

or,

$$\tilde{\boldsymbol{\theta}}(k+1) = \Sigma^{-1}(k+1)\Sigma(k)\tilde{\boldsymbol{\theta}}(k) - \Sigma^{-1}(k)\omega^T(k)\mathbf{L}(k)\tilde{\boldsymbol{\eta}}_y(k). \tag{5.42}$$

Using the parameter update law, the identifier matrix dynamics, the filter dynamics and the auxiliary variable dynamics, the rate change of the $V_{\tilde{\theta}}(k)$ is bounded as follows:

$$V_{\tilde{\theta}}(k+1) - V_{\tilde{\theta}}(k) \leq -(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k))^T\mathbf{L}(k)(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k)) + \tilde{\boldsymbol{\eta}}_y^T(k)\mathbf{L}(k)\tilde{\boldsymbol{\eta}}_y(k) \tag{5.43}$$

From the $\tilde{\boldsymbol{\eta}}(k)$ dynamics given in (5.9), it follows from Lemma 6, if $\bar{\mathbf{w}} \in l_2$ then $\tilde{\boldsymbol{\eta}}(k) \in l_2$. Taking the limit as $k \to \infty$, the inequality becomes

$$\begin{aligned} \lim_{k\to\infty} V_{\tilde{\theta}}(k) &= V_{\tilde{\theta}}(0) + \sum_{k=0}^{\infty} V_{\tilde{\theta}}(k+1) - V_{\tilde{\theta}}(k) \qquad (5.44) \\ &\leq V_{\tilde{\theta}}(0) - \sum_{k=0}^{\infty} \left[(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k))^T\mathbf{L}(k)(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k))\right] \\ &\quad + \sum_{k=0}^{\infty} \left[\tilde{\boldsymbol{\eta}}_y(k)^T\mathbf{L}(k)\tilde{\boldsymbol{\eta}}_y(k)\right]. \qquad (5.45) \end{aligned}$$

By the boundedness of the trajectories of the system, it follows that there exists a number $\gamma > 0$ such that

$$1 \geq \|\mathbf{L}(k)\| \geq \gamma.$$

as a result, one obtains the following inequality

$$\lim_{k\to\infty} V_{\tilde{\theta}}(k) \leq V_{\tilde{\theta}}(0) - \gamma \sum_{k=0}^{\infty} \left[(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k))^T (\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k)) \right] + \sum_{k=0}^{\infty} \left[\tilde{\boldsymbol{\eta}}_y(k)^T \tilde{\boldsymbol{\eta}}_y(k) \right] \tag{5.46}$$

Therefore, if conditions (5.39) are met, then the right hand side of (5.46) is finite. As a result, one concludes that:

$$\lim_{k\to\infty} \tilde{\boldsymbol{\theta}}(k) = 0 \tag{5.47}$$

as required. ■

Set Update

The following set update law guarantees the non exclusion of the true values of the estimates from the uncertainty set $B(\boldsymbol{\theta}_c, z_{\boldsymbol{\theta}_c})$:

$$z_{\hat{\boldsymbol{\theta}}}(k) = \sqrt{\frac{V_{z\hat{\boldsymbol{\theta}}}(k)}{4\lambda_{min}(\Sigma(k))}} \tag{5.48}$$

$$V_{z\boldsymbol{\theta}}(k+1) = V_{z\hat{\boldsymbol{\theta}}}(k) - (\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k))^T \mathbf{L}(k)(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k)) + \left(\frac{M_\vartheta}{K}\right)^2 \tag{5.49}$$

$$V_{z\hat{\boldsymbol{\theta}}}(0) = 4\lambda_{max}(\Sigma(0))(z_{\hat{\boldsymbol{\theta}}}(0))^2 \tag{5.50}$$

The set update approach can be summarized as the following algorithm.

Algorithm 5 *1. At time-step, $k = 0$, initialize $z_{\hat{\boldsymbol{\theta}}c} = z_{\hat{\boldsymbol{\theta}}}(0), \hat{\boldsymbol{\theta}}_c = \hat{\boldsymbol{\theta}}(0)$*

2. At time-step k, update:

$$(\hat{\boldsymbol{\theta}}_c, z_{\hat{\boldsymbol{\theta}}c}) = \begin{cases} (\hat{\boldsymbol{\theta}}(k), z_{\hat{\boldsymbol{\theta}}}(k)), & \text{if } \ z_{\hat{\boldsymbol{\theta}}}(k) \leq z_{\hat{\boldsymbol{\theta}}c} - ||\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}_c|| \\ (\hat{\boldsymbol{\theta}}_c, z_{\hat{\boldsymbol{\theta}}c}), & \text{otherwise} \end{cases}$$

3. Iterate back to step 2 and increment $k = k + 1$

The following lemma establishes the main properties of Algorithm 5.

Lemma 8 *Algorithm 5 ensures that*

1. *the set is only updated when updating will yield a contraction,*

2. *the dynamics of the set error bound described in (5.48) are such that they ensure the non-exclusion of the true value $\boldsymbol{\theta} \in \Theta(k)$, $\forall k$ if $\theta(0) \in \Theta(0)$.*

Proof:

1. If $\Theta(k+1) \nsubseteq \Theta(k)$ then

$$\sup_{s\epsilon\Theta(k+1)} \left\| s - \hat{\boldsymbol{\theta}}(k) \right\| \geq z_{\hat{\boldsymbol{\theta}}}(k) \tag{5.51}$$

However, it is guaranteed by the set update algorithm presented that Θ, at update times, obeys the following:

$$\sup_{s\epsilon\Theta(k+1)} \left\| s - \hat{\boldsymbol{\theta}}(k) \right\| \leq \sup_{s\epsilon\Theta(k+1)} \left\| s - \hat{\boldsymbol{\theta}}(k+1) \right\| + \left\| \hat{\boldsymbol{\theta}}(k+1) - \hat{\boldsymbol{\theta}}(k) \right\| \tag{5.52}$$

$$\leq z_{\hat{\boldsymbol{\theta}}}(k+1) + \left\| \hat{\boldsymbol{\theta}}(k+1) - \hat{\boldsymbol{\theta}}(k) \right\| \leq z_{\hat{\boldsymbol{\theta}}}(k) \tag{5.53}$$

This contradicts (5.51). Therefore, $\Theta(k+1) \subseteq \Theta(k)$ at time steps where Θ is updated.

2. It is known, by definition, that

$$V_{\tilde{\theta}}(0) \leq V_{z\theta}(0), \qquad \forall k \geq 0 \tag{5.54}$$

Since, $V_{\tilde{\boldsymbol{\theta}}}(k) = \tilde{\boldsymbol{\theta}}(k)^T \Sigma(k) \tilde{\boldsymbol{\theta}}(k)$,

$$\left\| \tilde{\theta}(k) \right\| \leq \frac{V_{z\hat{\theta}}(0)}{\lambda_{min}(\Sigma(k))} = 4z^2_{\hat{\theta}}(k), \qquad \forall k \geq 0 \tag{5.55}$$

Therefore, if $\theta(0)\ \epsilon\ \Theta(0)$, then $\theta(k)\ \epsilon\ \Theta(k)\ \ \forall\ k\ \geq\ 0$.

■

Simulation results

Test Problem

Consider the 3 state test problem with external disturbance:

$$\begin{aligned}
x_1(k+1) &= x_1(k) + 0.01(x_2(k) + x_3(k)^2\theta_1 - x_3(k)\theta_3) + \omega_1 \\
x_2(k+1) &= x_2(k) + 0.01(-x_1(k) + x_3(k) + x_3(k)\theta_2 + x_3(k)\theta_3) + \omega_2 \\
x_3(k+1) &= x_3(k) + 0.01(-x_1(k) - 2x_2(k) - x_3(k) + x_3(k)\theta_3) + \omega_3 \\
y &= x_3(k)
\end{aligned} \tag{5.56}$$

The unknown disturbance in given by:

$$\boldsymbol{\omega}(k) = \sin(k)[0.1 \quad 0.1 \quad 0.1]^T \tag{5.57}$$

The following estimates were used as initial conditions for the plant and estimator:

$$\begin{aligned}
\boldsymbol{\theta} &= [2.9 \;\; 3.1 \;\; 0.7]^T \\
\hat{\boldsymbol{\theta}}_c(0) &= [2.74 \;\; 3.18 \;\; 0.86]^T \\
z_\theta(0) &= 10; \\
\mathbf{x}(0) &= [0.1 \;\; 0.03 \;\; 0.04] \\
\hat{\mathbf{x}}(0) &= [0.4 \;\; 0.12 \;\; 0.16] \\
z_{\boldsymbol{\eta}}(0) &= 10.
\end{aligned}$$

In Figure 5.1 the state estimation error is showed. We notice that the error oscillate around the origin, following the inserted disturbance pattern. Under disturbance, the parameters true values are recovered, as displayed in Figure 5.3. The small offset in θ_1 and θ_2 can be decreased if a larger disturbance is inserted or a longer simulation is used. The parameter estimation error (Figure 5.4) and the auxiliary variable error (Figure 5.2) are bounded, leading to a bounded state estimation error.

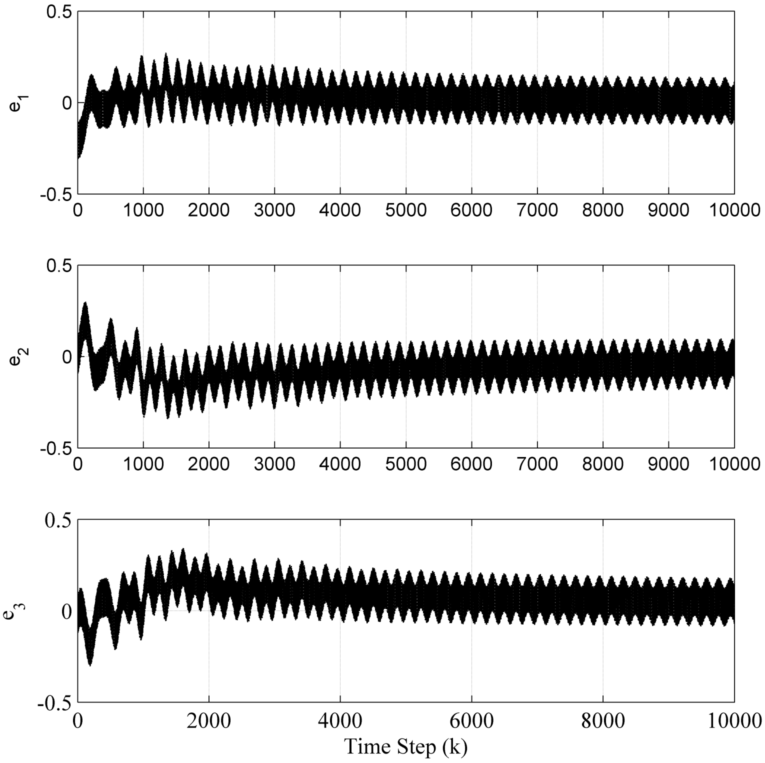

Figure 5.1: State estimation error $e(k) = x(k) - \hat{x}(k)$ along the simulation for the test problem.

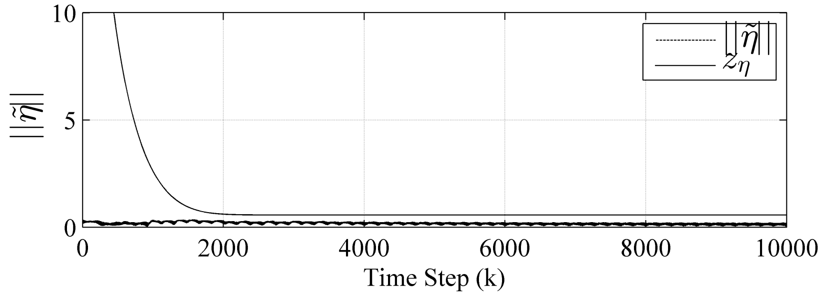

Figure 5.2: Uncertainty set radius z_{η} and the auxiliary variable norm along time.

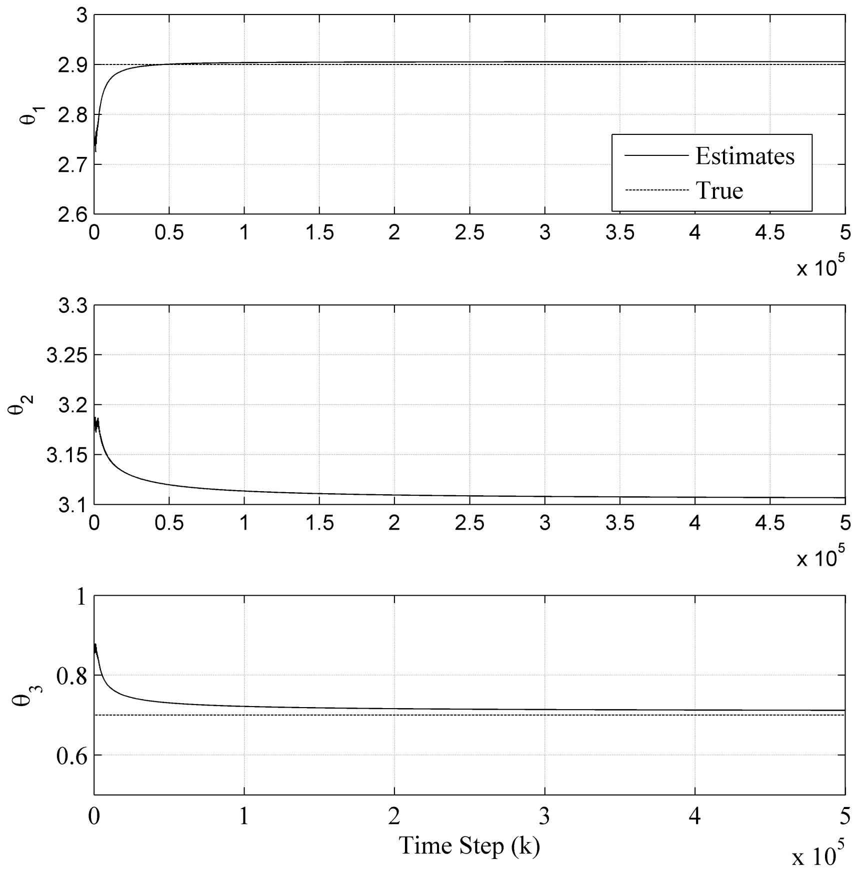

Figure 5.3: Parameter convergence for the test problem.

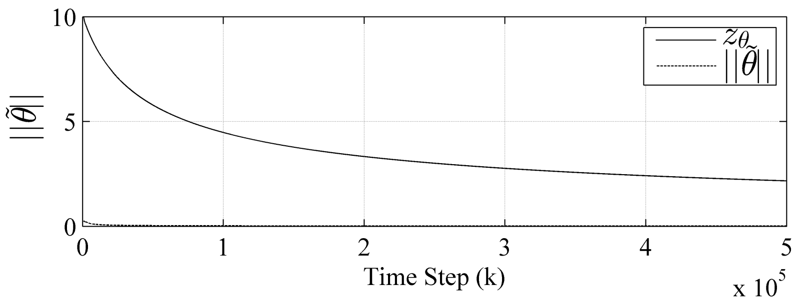

Figure 5.4: Parameter uncertainty set radius $z_{\boldsymbol{\theta}}$ and parameter estimation error along time.

Example 2

Consider an isothermal continuous stirred tank reactor (CSTR) with multi-component chemical reactions GHAFFARI *et al.* (2013):

$$A \rightleftharpoons B \longrightarrow C \tag{5.58}$$

The continuous-time model is represented by the following set of differential equations:

$$\begin{aligned}
\dot{\bar{x}}_1 &= 1 - \bar{x}_1 - Da_1\bar{x}_1 + Da_2\bar{x}_2^2 \\
\dot{\bar{x}}_2 &= Da_1\bar{x}_1 - \bar{x}_2 - Da_2\bar{x}_2^2 - Da_3\bar{x}_2^2 + \bar{u} \\
\dot{\bar{x}}_3 &= -\bar{x}_3 + Da_3\bar{x}_2^2
\end{aligned} \tag{5.59}$$

where $\bar{x}_i$ are dimensionless concentrations:

$$\bar{x}_i = \frac{C_i}{C_{AF}} \tag{5.60}$$

C_{AF} is the feed concentration of component A. Using deviation variables:

$$x_i = \bar{x}_i - x_{id} \tag{5.61}$$

$$u = \bar{u} - u_d = \frac{N_{BF}}{FC_{AF}} \tag{5.62}$$

Where N_{BF} is the B molar feeding rate and F the volumetric flow rate.

The differential equations can be rewritten as:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -(1+Da_1) & 2Da_2x_{2d} & 0 \\ Da_1 & -(1+2Da_2x_{2d}+2Da_3x_{2d}) & 0 \\ 0 & 2Da_3x_{2d} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} Da_2x_2^2 \\ -(Da_2+Da_3)x_2^2 \\ Da_3x_2^2 \end{bmatrix} \tag{5.63}$$

where $\mathbf{x}^T = [x_1, x_2, x_3]^T$ and $\mathbf{u} = u$. A first formulation (focused on the estimation of Da_2 and Da_3) provides a system belonging to the class 5.1 treated in this manuscript with a constant $\mathbf{A}$ and a matrix $\mathbf{B}(\mathbf{H}(\mathbf{x}(k)))$ given by (Case 1):

$$\mathbf{B}(\mathbf{H}(\mathbf{x}(k))) = \begin{bmatrix} x_2^2 & 0 \\ -x_2^2 & -x_2^2 \\ 0 & x_2^2 \end{bmatrix} \tag{5.64}$$

As one can notice, this formulation leads to an error in the observer, because the matrix multiplying $\mathbf{x}$ in (5.63) depends on the parameter vector. However, it is possible to avoid this error in the observer equations by transferring the linear terms to the $\mathbf{B}$ matrix, which leads to the following system (Case 2):

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -(1+Da_1) & 0 & 0 \\ Da_1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{u} + \underbrace{\begin{bmatrix} x_2^2 + 2x_{2d}x_2 & 0 \\ -x_2^2 - 2x_{2d}x_2 & -x_2^2 - 2x_{2d}x_2 \\ 0 & x_2^2 + 2x_{2d}x_2 \end{bmatrix}}_{\mathbf{B}(\mathbf{H}(\mathbf{x}(k)))} \begin{bmatrix} Da_2 \\ Da_3 \end{bmatrix} \tag{5.65}$$

The true value of the parameter vector is given by:

$$\boldsymbol{\theta} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \tag{5.66}$$

The initial estimates were:

$$\hat{\boldsymbol{\theta}}(0) = \begin{bmatrix} 0.4 \\ 1.3 \end{bmatrix} \tag{5.67}$$

The initial conditions for the state estimation routine and the plant were:

$$\begin{aligned}\hat{\mathbf{x}}(0) &= [-0.8, 0.8, 0.4] \\ \mathbf{x}(0) &= [-1, 1, 0.5]\end{aligned} \tag{5.68}$$

A set of step disturbances were included in the manipulated variable:

$$\begin{aligned}&u(k) = 1, \quad \text{for} \quad k \leq 5000 \\ &u(k) = 0, \quad \text{for} \quad 5000 < k \leq 10000 \\ &u(k) = 1, \quad \text{for} \quad 10000 < k \leq 20000 \\ &u(k) = 0, \quad \text{for} \quad 20000 < k \leq 30000 \\ &u(k) = -1, \quad \text{for} \quad 30000 < k \leq 40000 \\ &u(k) = 0, \quad \text{for} \quad 40000 < k \leq 50000\end{aligned} \tag{5.69}$$

The parameters used in the simulation were: $Da_1 = 3.0$, $Da_2 = 0.5$ and $Da_3 = 1.0$. The steady state values are given by $\mathbf{x}_d = [0.3467, 0.8796, 0.8796]$ and $u_d = 1.0$. The differential equations were discretized using the explicit Euler method with sampling time $T_s = 0.01$. The measurement matrix is given by:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.70}$$

The results of both cases were compared.

Figure 5.6 shows the time evolution of the state estimation error. In both cases, the state estimation error decreases as a function of time. However, the state estimation error decreases faster when (5.65) is used (Case 2). Since in the first formulation the true parameters values were used in matrix $\mathbf{A}$, the system parameters converge to the true values in both cases as shown in Figure 5.9. In Figure 5.9, the parameter θ_2 has its convergence influenced by the system excitation (as every parameter estimation routine), for the Case 1, since the model used for the estimation has a structural uncertainty, the parameter displays a step in the initial time-steps and then converges to the true value. The parameter error is bounded by the parameter uncertainty radius in both cases (Figures 5.5) and it decreases along time. In Figure 5.7, the error and the uncertainty radius for the auxiliary variable are shown. It can be seen that the final bound is smaller for the second case. Finally, the estimates for the states are displayed in Figure 5.8. The estimates of the unmeasured concentration (x_3) display no steady state offset in the second case, even as the parameter estimates are still updating.

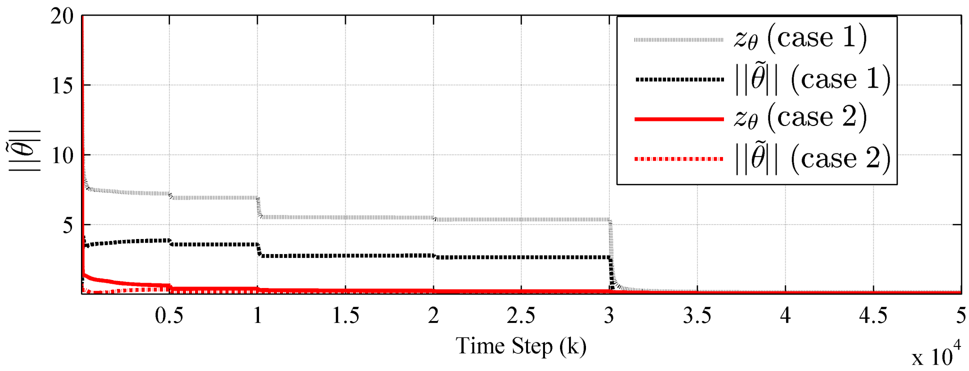

Figure 5.5: Progression of the parameter uncertainty set radius $z_{\boldsymbol{\theta}}$ and parameter error along time-step for the CSTR problem.

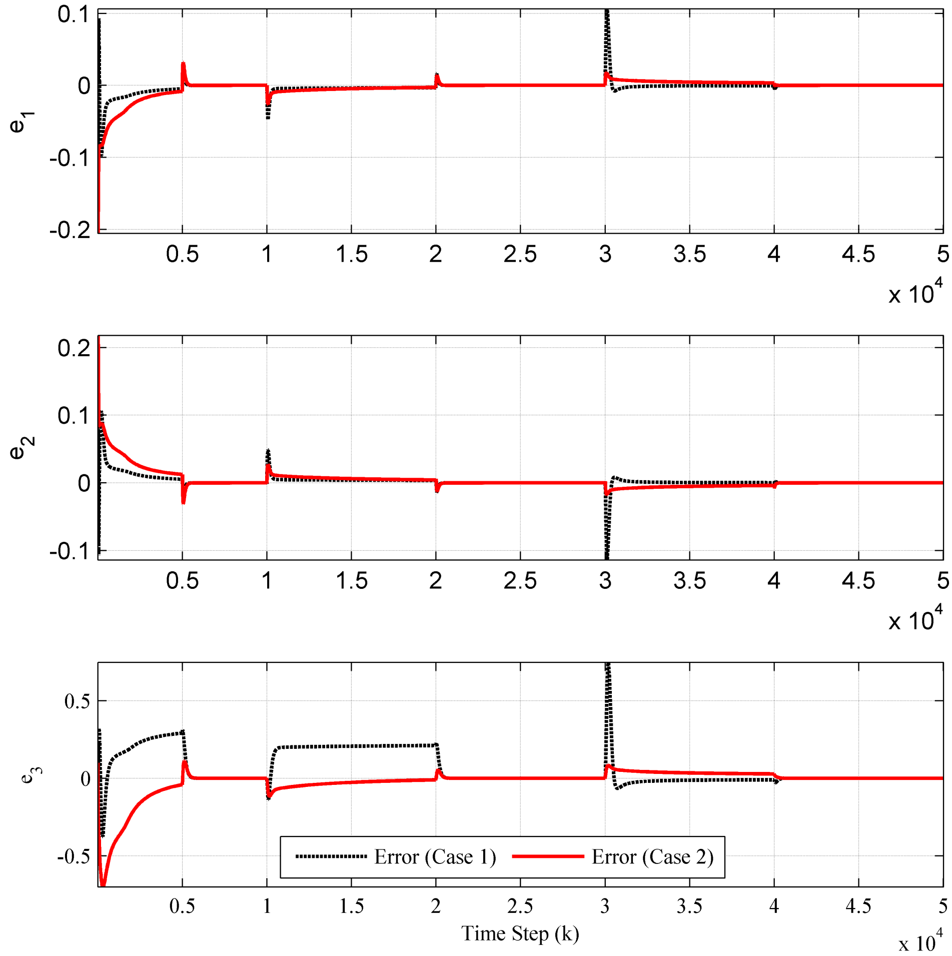

Figure 5.6: State estimation error $(\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k))$ as a function of time-step for the CSTR problem.

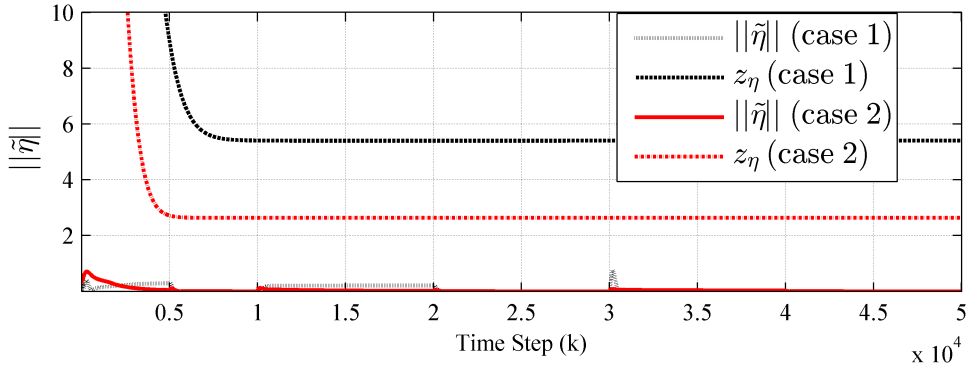

Figure 5.7: Auxiliary variable uncertainty radius z_η and auxiliary variable error

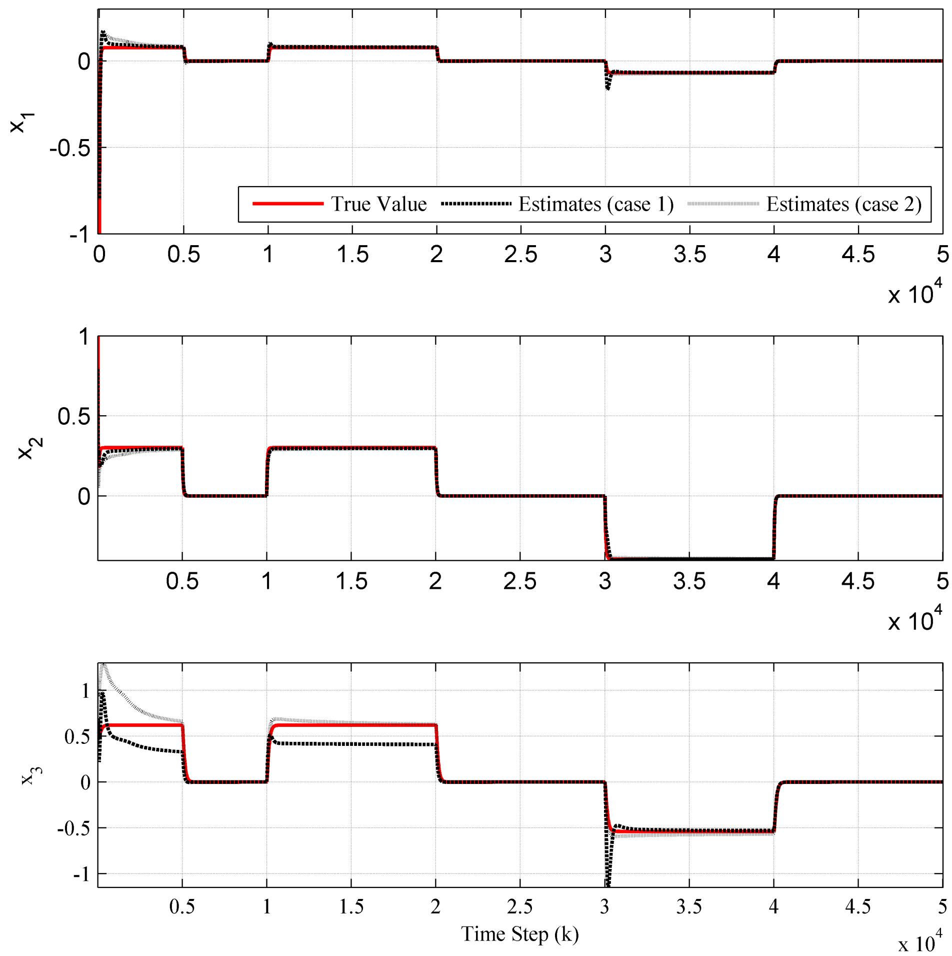

Figure 5.8: State estimates along time-step. Improvement in the estimates are observed while the parameters converge to the true values.

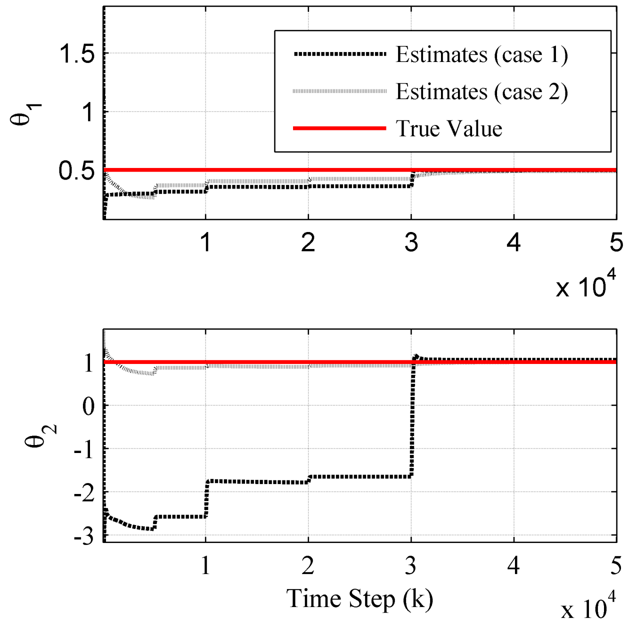

Figure 5.9: Parameter convergence along time-step.

5.2.2 Time-Varying Parameters

Problem description

Consider the previous nonlinear class with time-varying parameters:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{x}(k) + \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{y}(k))\boldsymbol{\theta}(k)\\ \mathbf{y}(k) &= \mathbf{H}\mathbf{x}(k)\end{aligned} \tag{5.71}$$

This system should satisfy the following assumptions:

Assumption 9 *The states are within the compact set* $\mathbb{X} \in \mathbb{R}^n$

Assumption 10 *The system is observable*

Assumption 11 *The time-varying dynamics of the parameters must satisfy* $\lim_{k\to\infty} \boldsymbol{\theta}(k) =$ *constant*

Assumption 12 *The parameters are initially within the known set* $\Theta^0 = B(\theta_0, z_\theta)$

The proposed estimator aims to provide state and uncertainty estimates ($\hat{\mathbf{x}}(k)$) by using discrete-time measurements ($\mathbf{y}(k)$). Moreover, the adaptive observer is able to update the equations parameters during the estimation process improving the state estimates.

State and Uncertainty Estimation

The following structure is proposed for the estimator:

$$\hat{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k) + \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}(\mathbf{y}(k))\hat{\boldsymbol{\theta}}(k+1) + \mathbf{K}(\mathbf{y}(k) - \mathbf{H}\hat{\mathbf{x}}(k))$$
$$- \mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) - (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) \quad (5.72)$$

Additionally, consider the following dynamics for the time-varying parameters:

$$\boldsymbol{\theta}(k) = \boldsymbol{\theta}_0 + \mu(k) \quad (5.73)$$

Assumption 13 *The time-varying component is unknown, however, it is bounded by* $\|\mu(k)\| \leq c$

$\mathbf{C}^T(k)$ is given by the following filter:

$$\mathbf{C}^T(k+1) = \mathbf{C}^T(k) + (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k) + \mathbf{B}(y(k)) \quad \mathbf{C}(0) = \mathbf{0} \quad (5.74)$$

the state-estimation error ($\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$) is written as:

$$\mathbf{e}(k+1) = \mathbf{e}(k) + (\mathbf{A} - \mathbf{KH})\mathbf{e}(k) + \mathbf{B}(\mathbf{y}(k))\tilde{\boldsymbol{\theta}}(k+1) + \mathbf{B}(\mathbf{y}(k))\mu(k)$$
$$+ \mathbf{C}^T(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) + (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) \quad (5.75)$$

Additionally, consider an auxiliary variable defined as:

$$\boldsymbol{\eta}(k+1) = \mathbf{e}(k+1) - \mathbf{C}^T(k+1)\tilde{\boldsymbol{\theta}}(k+1) \quad (5.76)$$

Combining (5.75) and (5.74), the auxiliary variable dynamics is given by:

$$\boldsymbol{\eta}(k+1) = \boldsymbol{\eta}(k) + (\mathbf{A} - \mathbf{KH})\boldsymbol{\eta}(k) + \mathbf{B}(\mathbf{y}(k))\mu(k) \quad (5.77)$$

Since the time-varying part of the parameter dynamic is unknown, an estimate for the auxiliary variable dynamics is given by:

$$\hat{\boldsymbol{\eta}}(k+1) = \hat{\boldsymbol{\eta}}(k) + (\mathbf{A} - \mathbf{KH})\hat{\boldsymbol{\eta}}(k) \quad (5.78)$$

leading to the error:

$$\tilde{\boldsymbol{\eta}}(k+1) = \underbrace{(\mathbf{I} + \mathbf{A} - \mathbf{KH})}_{\tilde{\mathbf{A}}}\tilde{\boldsymbol{\eta}}(k) + \mathbf{B}(\mathbf{y}(k))\mu(k) \quad (5.79)$$

The computation of (5.79) is not possible. However, an upper bound can be obtained by the Lyapunov function candidate:

$$V(k) = \tilde{\boldsymbol{\eta}}(k)^T \mathbf{P} \tilde{\boldsymbol{\eta}}(k) \tag{5.80}$$

By substitution of (5.79) in (5.80), and computing the time variation of the candidate function, it follows that:

$$\begin{aligned} V(k+1) - V(k) &= (\tilde{\mathbf{A}}\tilde{\boldsymbol{\eta}}(k) + \mathbf{B}(\mathbf{y}(k))\mu(k))^T \mathbf{P} (\tilde{\mathbf{A}}\tilde{\boldsymbol{\eta}}(k) + \mathbf{B}(\mathbf{y}(k))\mu(k)) - \tilde{\boldsymbol{\eta}}(k)^T \mathbf{P} \tilde{\boldsymbol{\eta}}(k) \\ &= \tilde{\boldsymbol{\eta}}(k)^T \underbrace{(\tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}} - \mathbf{P})}_{-\bar{\mathbf{Q}}} \tilde{\boldsymbol{\eta}}(k) + \tilde{\boldsymbol{\eta}}(k)^T \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) \\ &\quad + \mu(k)^T \mathbf{B}(\mathbf{y}(k))^T \mathbf{P} \tilde{\mathbf{A}} \tilde{\boldsymbol{\eta}}(k) + \mu(k)^T(k) \mathbf{B}(\mathbf{y}(k))^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) \end{aligned} \tag{5.81}$$

Since $\mathbf{P}$ is symmetric and positive definite:

$$\begin{aligned} V(k+1) - V(k) = -\tilde{\boldsymbol{\eta}}(k)^T \bar{\mathbf{Q}} \tilde{\boldsymbol{\eta}}(k) + 2\tilde{\boldsymbol{\eta}}(k)^T \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) + \\ \mu(k)^T(k) \mathbf{B}(\mathbf{y}(k))^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) \end{aligned} \tag{5.82}$$

The following inequalities can be used to bound each term of the Lyapunov function variation. The first term can be bounded as follows:

$$\tilde{\boldsymbol{\eta}}(k)^T \bar{\mathbf{Q}} \tilde{\boldsymbol{\eta}}(k) \geq \lambda_{min}(\mathbf{Q}) \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) \geq \frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} V(k). \tag{5.83}$$

The second term yields:

$$\begin{aligned} \tilde{\boldsymbol{\eta}}(k)^T \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) &\leq \frac{1}{2} \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) + \frac{1}{2} \mu(k)^T \mathbf{B}(\mathbf{y}(k))^T \mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) \\ &\leq \frac{1}{2\lambda_{min}(\mathbf{P})} V(k) + \frac{||\mathbf{B}(\mathbf{y}(k))^T \mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))||}{2} c, \end{aligned} \tag{5.84}$$

and, the third term:

$$\mu(k)^T(k) \mathbf{B}(\mathbf{y}(k))^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))\mu(k) \leq \lambda_{max}(\mathbf{P}) ||\mathbf{B}(\mathbf{y}(k))|| c. \tag{5.85}$$

As a result, one obtains:

$$V(k+1) - V(k) \leq -\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2\lambda_{min}(\mathbf{P})}\right)V(k) \\ + \frac{||\mathbf{B}(\mathbf{y}(k))^T\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}\mathbf{B}(\mathbf{y}(k))||}{2}c + \lambda_{max}(\mathbf{P})||\mathbf{B}(\mathbf{y}(k))||c \tag{5.86}$$

Consider the following equations for the recursive update of the uncertainty set:

$$\begin{aligned} z_{\boldsymbol{\eta}}(k) =& \sqrt{\frac{V_{z\eta}(k)}{\lambda_{min}(\mathbf{P})}} \\ V_{z\boldsymbol{\eta}}(0) =& \lambda_{max}(\mathbf{P})z_{\boldsymbol{\eta}}(0)^2 \\ V_{z\boldsymbol{\eta}}(k+1) =& V_{z\boldsymbol{\eta}}(k) - \left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2\lambda_{min}(\mathbf{P})}\right)V(k) \\ &+ \left(\frac{||\mathbf{B}(\mathbf{y}(k))^T\mathbf{P}^T\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\mathbf{P}\mathbf{B}(\mathbf{y}(k))||}{2} + \lambda_{max}(\mathbf{P})||\mathbf{B}(\mathbf{y}(k))||\right)c \end{aligned} \tag{5.87}$$

Additionally, the update follows the algorithm 6:

Algorithm 6 *Uncertainty Set Update* $\mathcal{X}(k) \triangleq B(0, z_{\boldsymbol{\eta}}(k))$

1. *If* $k = 0$, $z_{\boldsymbol{\eta}}(0) = z_{\boldsymbol{\eta}0}$

2. *At* k, *update:*

$$\mathcal{X}(k+1) = \begin{cases} B\left(0, z_{\boldsymbol{\eta}}(k)\right), & \textit{if } z_{\boldsymbol{\eta}}(k) \leq z_{\boldsymbol{\eta}}(k-1) \\ B\left(0, z_{\boldsymbol{\eta}}(k-1)\right), & \textit{otherwise} \end{cases} \tag{5.88}$$

3. $k = k+1$, *return to step 2*

Lemma 9 *If Equations 5.87 and Algorithm 6 are used, then the set evolution* $\mathcal{X}(k) \triangleq B(0, z_{\boldsymbol{\eta}}(k))$ *is such that:*

1. $\tilde{\eta}(0) \in \mathcal{X}(0) \to \tilde{\eta}(k) \in \mathcal{X}(k)$

2. $\mathcal{X}(k+1) \subseteq \mathcal{X}(k)$

Proof:

1. *By definition:* $V(0) \leq V_{z\boldsymbol{\eta}}(0)$. *Furthermore* $\Delta V(k) \leq \Delta V_{z\boldsymbol{\eta}}(k)$, *which leads to:*

$$V(k) \leq V_{z\boldsymbol{\eta}}(k) \quad \forall k \geq 0 \tag{5.89}$$

The Lyapunov function candidate $V(k)$ is such that:

$$V(k) = \tilde{\boldsymbol{\eta}}(k)^T \mathbf{P} \tilde{\boldsymbol{\eta}}(k) \geq \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) \lambda_{min}(\mathbf{P})$$

As a result, one can write:

$$\frac{V(k)}{\lambda_{min}(\mathbf{P})} \geq \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k)$$

which ensures that:

$$z_{\boldsymbol{\eta}}^2(k) = \frac{V_{z\boldsymbol{\eta}}(k)}{\lambda_{min}(\mathbf{P})} \geq \tilde{\boldsymbol{\eta}}(k)^T \tilde{\boldsymbol{\eta}}(k) \quad \forall k \geq 0$$

This last inequality implies that if $\boldsymbol{\eta} \in \mathcal{X}(0)$, then $\boldsymbol{\eta} \in B(\hat{\boldsymbol{\eta}}(k), z_{\boldsymbol{\eta}}(k)) \ \ \forall k \geq 0$, as required.

2. *If $\mathcal{X}(k+1) \nsubseteq \mathcal{X}(k)$ then:*

$$\sup_{\tilde{\boldsymbol{\eta}} \in \mathcal{X}(k+1)} ||\tilde{\boldsymbol{\eta}}(k)|| \geq \mathbf{z}_{\boldsymbol{\eta}}(k) \tag{5.90}$$

In order to have $\Delta V_{z\boldsymbol{\eta}}(k) \leq 0$:

$$V_{z\boldsymbol{\eta}}(k) \geq \frac{\left(\frac{||\mathbf{B}(\mathbf{y}(k))^T \mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))||}{2} + \lambda_{max}(\mathbf{P}) ||\mathbf{B}(\mathbf{y}(k))||\right) c}{\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2\lambda_{min}(\mathbf{P})}\right)} \tag{5.91}$$

At time-step $k+1$:

$$V_{z\boldsymbol{\eta}}(k+1) \leq \max\left[V_{z\boldsymbol{\eta}}(k), \frac{\left(\frac{||\mathbf{B}(\mathbf{y}(k))^T \mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))||}{2} + \lambda_{max}(\mathbf{P}) ||\mathbf{B}(\mathbf{y}(k))||\right) c}{\left(\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2\lambda_{min}(\mathbf{P})}\right)}\right] \tag{5.92}$$

$$z_{\boldsymbol{\eta}}(k+1) \leq \max\left[z_{\boldsymbol{\eta}}(k), \sqrt{\frac{\left(\frac{||\mathbf{B}(\mathbf{y}(k))^T \mathbf{P}^T \tilde{\mathbf{A}} \tilde{\mathbf{A}}^T \mathbf{P} \mathbf{B}(\mathbf{y}(k))||}{2} + \lambda_{max}(\mathbf{P}) ||\mathbf{B}(\mathbf{y}(k))||\right) c}{\left(\frac{\lambda_{min}(\mathbf{Q})\lambda_{min}(\mathbf{P})}{\lambda_{max}(\mathbf{P})} - \frac{1}{2}\right)}}\right] \tag{5.93}$$

Since the set is only updated if we have a set contraction ($z_{\boldsymbol{\eta}}(k+1) \leq z_{\boldsymbol{\eta}}(k)$), it follows that:

$$\sup_{\tilde{\boldsymbol{\eta}} \in \mathcal{X}(k+1)} ||\tilde{\boldsymbol{\eta}}(k)|| \leq \mathbf{z}_{\boldsymbol{\eta}}(k+1) \leq \mathbf{z}_{\boldsymbol{\eta}}(k), \tag{5.94}$$

a contradiction of (5.90)*. This completes the proof.*

■

Example 3

Consider the continuous stirred tank reactor (CSTR) used in Subsection 5.2.1. However, in this simulation the true value of the parameter vector is given by the following time varying functions:

$$\boldsymbol{\theta}(k) = \begin{bmatrix} 0.5 - 0.1(1 - e^{\frac{-k \cdot T_s}{\tau}}) \\ 1 - 0.1(1 - e^{\frac{-k \cdot T_s}{\tau}}) \end{bmatrix} \tag{5.95}$$

The true value of the fixed parameter is $Da_1 = 3.0$; the steady state values are given by $\mathbf{x}_d = [0.3467, 0.8796, 0.8796]$; the time constant for the parameter function is $\tau = 10$. The differential equations were discretized using the explicit Euler method with sampling time $T_s = 0.01$. The measurement matrix is given by:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.96}$$

Figure 5.10 shows the state estimation error. The convergence to an origin neighbourhood occurs in the first time-steps. Furthermore, the state estimates recover the real state values quickly, as shown in Figure 5.12. The time-varying parameters decrease along time and the same pattern is recovered by the parameter estimates (Figure 5.14). Moreover, the uncertainty set for the parameters decreases along time, providing a worst-case bound for the parameter error (Figure 5.11). Finally, the uncertainty set radius for the auxiliary variable, initially very large ($z_{\boldsymbol{\eta}} = 10$), decreases with time, achieving a tight bound on the error on the variable ($\tilde{\boldsymbol{\eta}}(k)$) and, correspondingly, on the state estimation error.

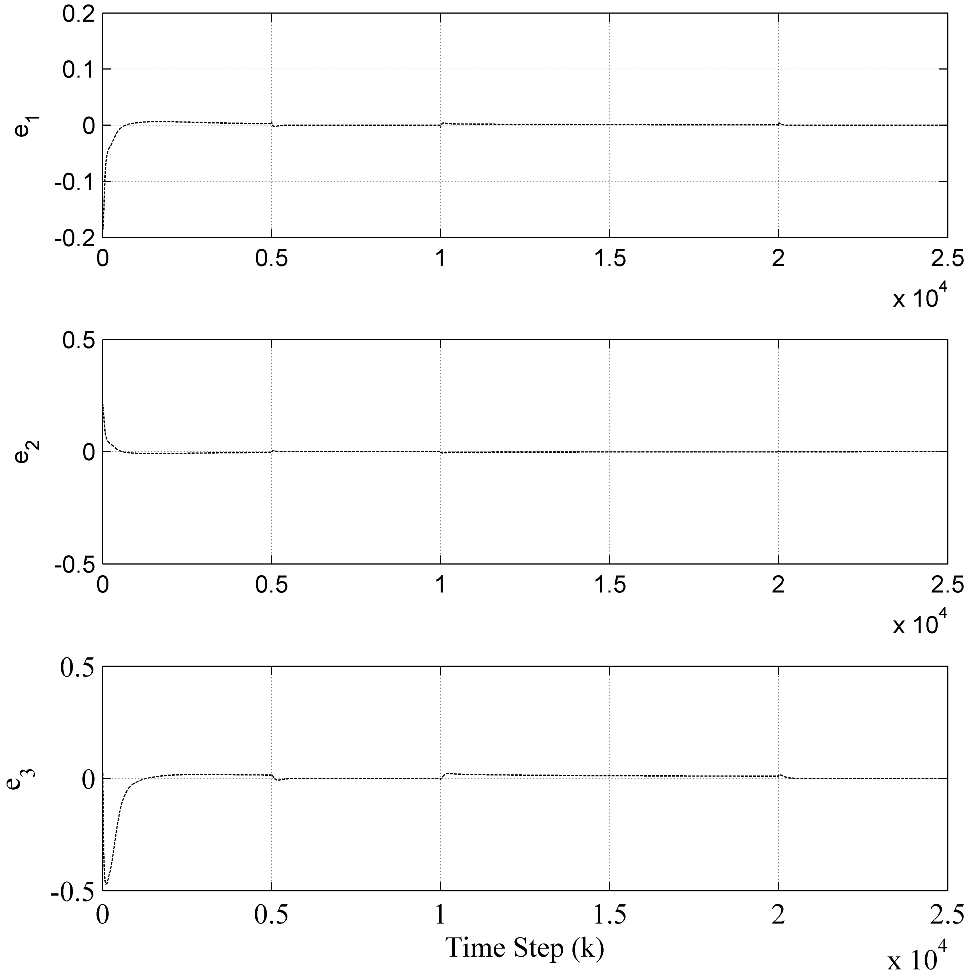

Figure 5.10: State estimation error ($\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$) as a function of time-step for the CSTR problem using time-varying parameters.

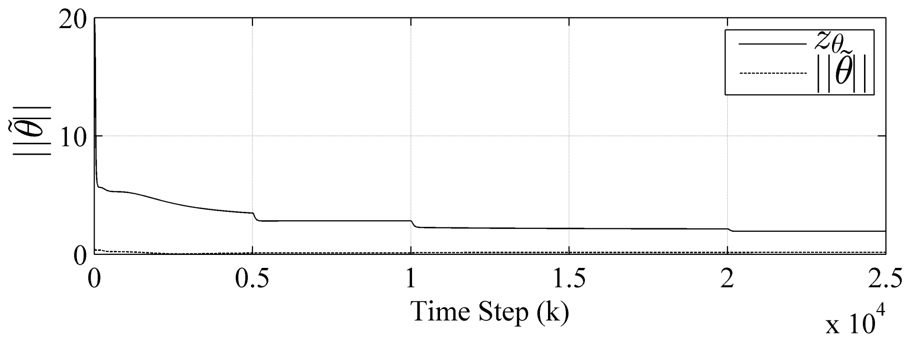

Figure 5.11: Progression of the parameter uncertainty set radius $z_{\boldsymbol{\theta}}$ and parameter error along time-step for the CSTR problem using time-varying parameters.

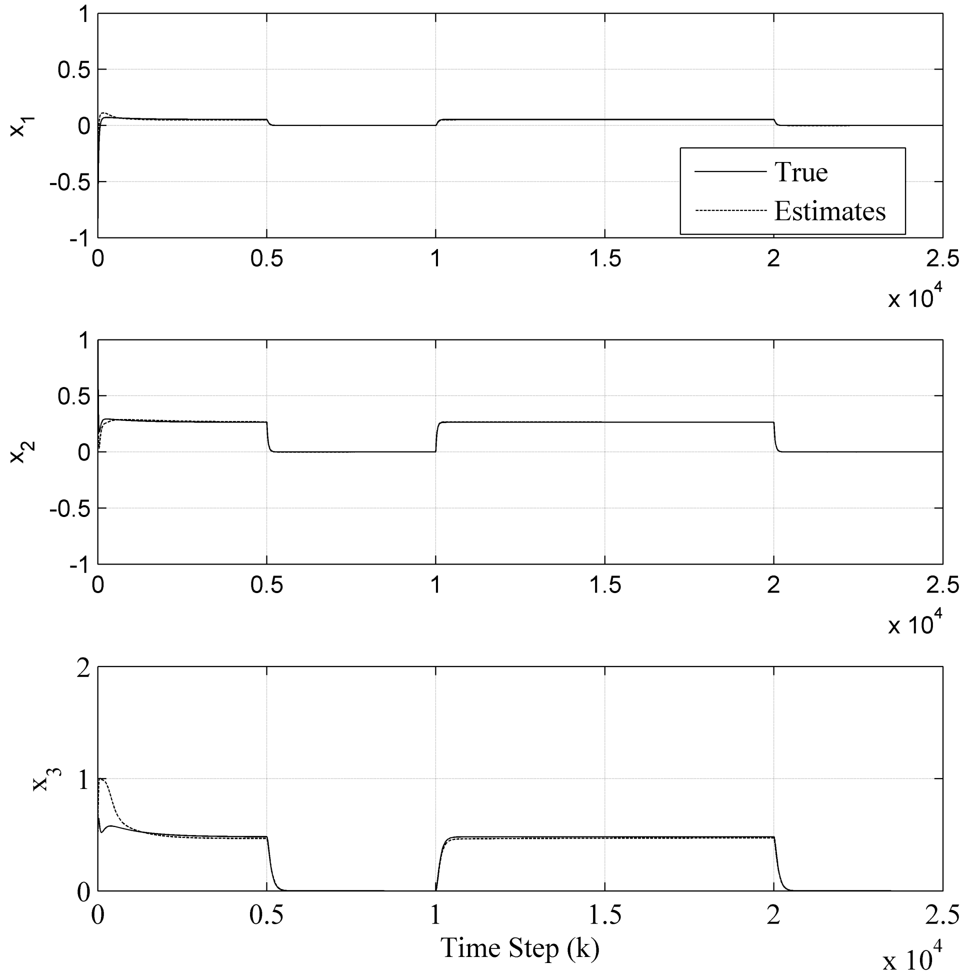

Figure 5.12: State estimates along time-step (time-varying parameters case).

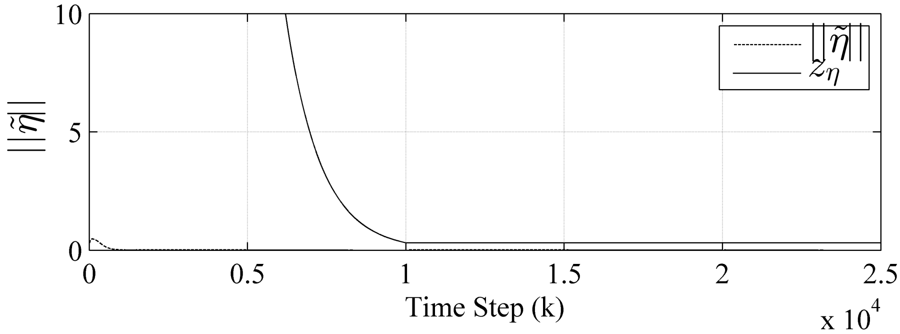

Figure 5.13: Plot of the auxiliary variable uncertainty radius z_{η} and auxiliary variable error along time-step (time-varying parameters case).

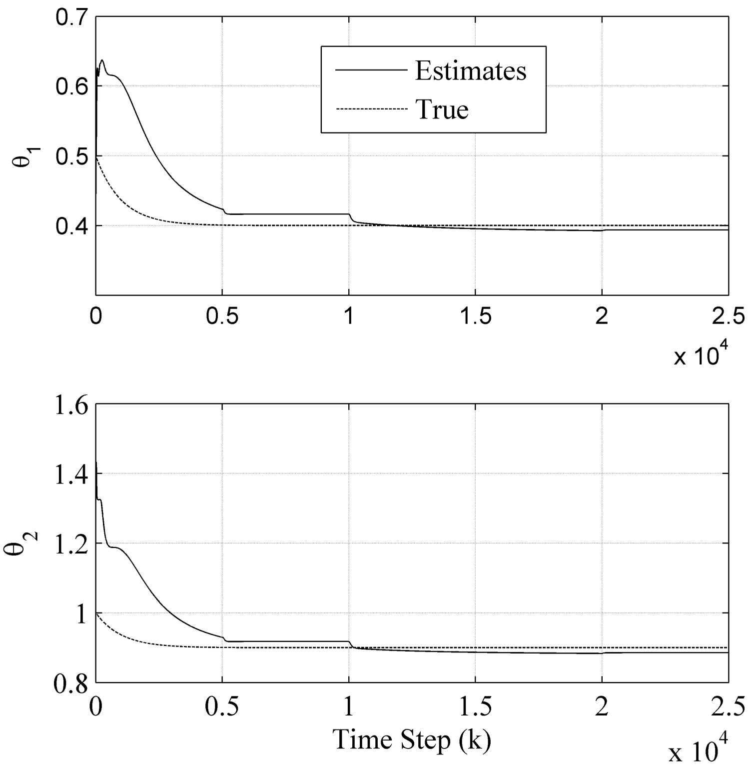

Figure 5.14: Parameter convergence along time-step using time-varying parameters.

5.3 Discrete-time adaptive MPC by Output Feedback

In this section, an output feedback nonlinear model predictive control is proposed which incorporates the set-based adaptive observers described in Section 2. In output feedback nonlinear MPC, a dynamic optimization problem is solved in real-time to compute the optimal input trajectories. The optimization problem relies on the availability of the current values of the state variables as an estimate for the future model predictions. In the absence of full state measurements, state estimates must be used to provide this initial condition. Since the state estimates will be not provide instantaneous agreement with the true state values, model uncertainties associated with poor state estimates may lead to constraint violation. If one considers the practical impossibility of a true model and the inevitable time-varying component arising in many practical problems, the control action using a fixed model is always suboptimal and, possibly, infeasible.

A way to improve the suboptimal solution is to consider explicitly the uncertainty in a robust optimization problem. Although this may lead to suboptimal performance, the question of feasibility can be treated adequately. Two practical problems arise in robust MPC formulations. The first one is to obtain an uncertainty description that can be computed in a real-time framework. Once the uncertainty representation is obtained, the second problem is to solve the nonlinear robust problem, which is not computable online, except for simple problems. In the present approach, we use the set-based estimation to solve the uncertainty problem in a real-time approach. Furthermore, the min-max problem is replaced by a Lipschitz-based MPC guaranteeing an over-approximation for the reachable sets and allowing the robust problem solution for a nonlinear model.

Using the previous approach for state and parameter estimation, it is possible to obtain a computable uncertanity bound that can be used explicitly in the MPC formulation. By construction, using the equations of Section 5.2.1, it is possible to obtain a set for the state estimates:

$$\mathbf{e}(k) = \boldsymbol{\eta}(k) + \mathbf{C}^T(k)\tilde{\boldsymbol{\theta}}(k). \tag{5.97}$$

The state estimation error is bounded as:

$$||\mathbf{e}(k)|| \leq ||\boldsymbol{\eta}(k)|| + ||\mathbf{C}^T(k)||||\tilde{\boldsymbol{\theta}}(k)|| \tag{5.98}$$

Using: $\boldsymbol{\eta} = \tilde{\boldsymbol{\eta}} + \hat{\boldsymbol{\eta}}$ and $\tilde{\boldsymbol{\theta}}^T(k)\tilde{\boldsymbol{\theta}}(k) = 4z_{\hat{\boldsymbol{\theta}}}^2(k)$

$$\begin{aligned} ||\mathbf{e}(k)|| &\leq ||\tilde{\boldsymbol{\eta}}(k)|| + ||\hat{\boldsymbol{\eta}}(k)|| + ||\mathbf{C}^T(k)||2z_{\hat{\boldsymbol{\theta}}}(k) \\ ||\mathbf{e}(k)|| &\leq z_{\boldsymbol{\eta}} + ||\hat{\boldsymbol{\eta}}(k)|| + ||\mathbf{C}^T(k)||2z_{\hat{\boldsymbol{\theta}}}(k) \end{aligned} \tag{5.99}$$

From the results in Section 2, it follows that the bound (5.99) is computable. As a result, one can guarantee that $\mathbf{e}(k) \in B(0, z_e(k))$ for $z_e(k) = z_{\boldsymbol{\eta}}(k) + ||\hat{\boldsymbol{\eta}}(k)|| + 2||\mathbf{C}^T(k)||z_{\hat{\boldsymbol{\theta}}}(k)$ and

$$\mathbf{x}(k) \in B\left(\hat{\mathbf{x}}(k), \frac{z_e(k)}{2}\right) \triangleq \bar{\mathcal{X}}(k). \tag{5.100}$$

Therefore, the use of the proposed set-based approach for joint estimation can guarantee an uncertainty estimate for the states suitable for robust MPC design. This approach is presented in the next section.

5.3.1 A Min-max Approach

Consider the variables and the sets: $\boldsymbol{\theta} \in \Theta$, $\vartheta \in \mathcal{D}$ and $\mathbf{x} \in \bar{\mathcal{X}}$. The variables $\boldsymbol{\theta}$ are the system parameters, ϑ are bounded disturbances and, $\mathbf{x}$, the state variables.

Let the control law, κ, be defined as follows:

$$u = \kappa_{mpc}(\hat{\mathbf{x}}, \hat{\theta}, z_\theta, z_x) \triangleq \kappa^*(0, \hat{\mathbf{x}}, \hat{\theta}, z_\theta, z_x) \tag{5.101a}$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot,\cdot,\cdot,\cdot)} J(\hat{\mathbf{x}}, \hat{\theta}, z_\theta, z_x, \kappa) \tag{5.101b}$$

where

$$J(\hat{\mathbf{x}}, \hat{\boldsymbol{\theta}}, z_\theta, z_x, \kappa) \triangleq \max_{\theta\in\Theta,\ \vartheta\in\mathcal{D}, \mathbf{x}\in\bar{\mathcal{X}}} \sum_{k=0}^{T-1} L(\mathbf{x}(k)^p, \mathbf{u}(k)^p) + W(\mathbf{x}(T)^p, \tilde{\boldsymbol{\theta}}(T)^p, z_x^p) \tag{5.102a}$$

$s.t.\ \ \forall k \in [0,\, T]$, where T is the prediction horizon

$$\mathbf{x}(k+1)^p = \mathbf{x}(k) + \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{x}(k)^p, \mathbf{u}(k)^p)\theta + \vartheta(k), \ \ \mathbf{x}_0^p = \hat{\mathbf{x}}(0) \tag{5.102b}$$

$$\begin{aligned}\hat{\mathbf{x}}(k+1) &= \hat{\mathbf{x}}(k) + \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}(\mathbf{y}(k), \mathbf{u}(k))\hat{\boldsymbol{\theta}}(k+1) + \mathbf{K}(\mathbf{y}(k) - \mathbf{H}\hat{\mathbf{x}}(k)) \\ &-\mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1)) - (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k)(\hat{\boldsymbol{\theta}}(k) - \hat{\boldsymbol{\theta}}(k+1))\end{aligned} \tag{5.102c}$$

$$\mathbf{C}^T(k+1) = \mathbf{C}^T(k) + (\mathbf{A} - \mathbf{KH})\mathbf{C}^T(k) + \mathbf{B}(y(k), \mathbf{u}(k)) \tag{5.102d}$$

$$\hat{\boldsymbol{\theta}}^p(k+1) = \hat{\boldsymbol{\theta}}^p(k) + (\Sigma^{-1}(k))^p\omega^T(k)\mathbf{L}(k)(\mathbf{e}_y(k) - \hat{\boldsymbol{\eta}}_y(k)) \tag{5.102e}$$

$$\bar{\hat{\boldsymbol{\theta}}}^p(k+1) = \mathrm{Proj}\{\hat{\boldsymbol{\theta}}^p(k+1), \Theta\} \tag{5.102f}$$

$$\tilde{\boldsymbol{\theta}}^p = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k^p, \ \ \hat{\boldsymbol{\theta}}_0^p = \hat{\boldsymbol{\theta}} \tag{5.102g}$$

$$\mathbf{u}^p(\tau) \triangleq \kappa(\tau, \mathbf{x}^p(\tau), \hat{\boldsymbol{\theta}}^p(\tau)) \in \mathbb{U} \tag{5.102h}$$

$$\mathbf{x}^p(\tau) \in \mathbb{X}, \ \ \mathbf{x}^p(T) \in \mathbb{X}_f(\tilde{\boldsymbol{\theta}}^p(T), z_x) \tag{5.102i}$$

The particular feature of this MPC formulation is that the effect of future parameter adaptation is incorporated as a constraint, (5.102c)-(5.102f), in the dynamic optimization problem.

The conservativeness of the algorithm is reduced by parameterizing both W and $\mathbb{X}_f$ as functions of $\tilde{\theta}(T)$ and the predicted state uncertainty $z_x(T)$. While it is possible for the sets Θ and $\mathcal{X}$ to contract over time, the robustness feature due to $\vartheta \in \mathcal{D}$ is preserved. The resulting algorithm (7) provides a robust output-feedback nonlinear MPC design method.

Algorithm 7 *The MPC algorithm performs as follows: At sampling instant k*

1. ***Measure*** *the current output of the plant* $\mathbf{y}_k$ *and obtain the current value of matrices* $\mathbf{C}(k)$*,* $w(k)$ *and* Σ^{-1} *from equations 5.4, 5.32 and 5.35 respectively*

2. ***Obtain*** *the current value of parameter estimates* $\hat{\boldsymbol{\theta}}$*, state estimates* $\hat{\mathbf{x}}$ *and uncertainty bounds* z_θ *and* $z_{\boldsymbol{\eta}}$ *from algorithms 5, and 6 respectively.*

3. ***Solve*** *the optimization problem* (5.101) *and apply the resulting feedback control law to the plant until the next sampling instant*

4. ***Increment*** $k = k+1$*. Repeat the procedure from step 1 for the next sampling instant.*

Despite the robustness guaranteed by the min-max approach, its solution remains impractical for real-time applications. A Lipschitz-based approach is proposed in the next section to alleviate the computational challenges associated with the min-max approach design formulation.

5.3.2 Lipschitz-based Approach

In this section, we present a Lipschitz-based method whereby the nominal model rather than the unknown bounded system state is controlled, subject to conditions that ensure constraint satisfaction for all possible uncertainties. State prediction error bound is determined based on the Lipschitz continuity of the model. A knowledge of appropriate Lipschitz bounds for the y-dependence of the $\mathbf{B}(y(k), u(k))$ dynamics is required.

Assumption 14 *A set of functions $\mathcal{L}_j : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^+$, $j \in \{\mathbf{B}(y(k), u(k))\}$ is known which satisfies*

$$\mathcal{L}_j(\mathbb{X}, u) \geq \min \left\{ \mathcal{L}_j \,\middle|\, \sup_{x_1, x_2 \in \mathbb{X}} \Big(\|j(x_1, u) - j(x_2, u)\| - \mathcal{L}_j \|x_1 - x_2\| \Big) \leq 0 \right\}.$$

This is an induced norm since $\mathbf{B}(y(k), u(k))$ is a matrix.

Assuming a knowledge of the Lipschitz bounds for the y-dependence of the dynamics $\mathbf{B}(y(k), u(k))$ as given in Assumption 14, the plant can be described by:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{x}(k) + \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{y}(k), \mathbf{u}(k))\boldsymbol{\theta} \\ \mathbf{y}(k) &= \mathbf{H}\mathbf{x}(k) \end{aligned} \tag{5.103}$$

The prediction model is thus given by:

$$\begin{aligned} \mathbf{x}_p(k+1) &= \mathbf{x}_p(k) + \mathbf{A}\mathbf{x}_p(k) + \mathbf{B}(\mathbf{y}(k), \mathbf{u}(k))\hat{\boldsymbol{\theta}} + \mathbf{w}(k) \\ \mathbf{y}(k) &= \mathbf{H}\mathbf{x}_p(k) + \mathbf{v}(k) \end{aligned} \tag{5.104}$$

A worst-case deviation $z_x^p(k) \geq \max_{\theta \in \Theta} \|x(k) - x(k)^p\|$ can be generated from:

$$\begin{aligned} \mathbf{x}(k+1) - \mathbf{x}_p(k+1) = \mathbf{A}(\mathbf{x}(k) - \mathbf{x}_p(k)) & \\ + \mathbf{B}(\mathbf{H}\mathbf{x}(k), \mathbf{u}(k))\boldsymbol{\theta} - \mathbf{B}(\mathbf{H}\mathbf{x}_p(k) + \mathbf{v}(k), \mathbf{u}(k))\boldsymbol{\theta} & \\ + \mathbf{B}(\mathbf{H}\mathbf{x}_p(k) + \mathbf{v}(k), \mathbf{u}(k))(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \mathbf{w} & \end{aligned} \tag{5.105}$$

Using the Lipschitz constant for $\mathbf{B}(\mathbf{y}(k), \mathbf{u}(k))$, we have:

$$\begin{aligned}\underbrace{||\mathbf{x}(k+1) - \mathbf{x}_p(k+1)||}_{z_x^p(k+1)} \leq ||A|| \underbrace{||\mathbf{x}(k) - \mathbf{x}_p(k)||}_{z_x(k)} \\ + \mathcal{L}_B ||\boldsymbol{\theta}|| ||\mathbf{H}|| \underbrace{||\mathbf{x}(k) - \mathbf{x}_p(k)||}_{z_x^p(k)} + \mathcal{L}_B ||\boldsymbol{\theta}|| ||\mathbf{H}|| ||\mathbf{v}|| \\ + ||\mathbf{B}(\mathbf{H}\mathbf{x}_p(k) + \mathbf{v}(k), \mathbf{u}(k))|| \underbrace{||(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(k))||}_{\leq z_\theta(k)} + ||\mathbf{w}||\end{aligned} \tag{5.106}$$

Thus, a worst case prediction of the uncertainty, in time-step k, is given by:

$$z_x^p(k+1) = (||\mathbf{A}|| + \mathcal{L}_B \Pi ||\mathbf{H}||) z_x^p(k) + ||\mathbf{B}(\mathbf{y}(k), \mathbf{u}(k))|| z_{\boldsymbol{\theta}}(k) + M_v \tag{5.107}$$

where:

$$\Pi = ||\hat{\boldsymbol{\theta}}|| + z_\theta(k) \tag{5.108}$$

$$M_v = ||\mathbf{w}|| + \mathcal{L}_B \Pi ||\mathbf{H}|| ||\mathbf{v}|| \tag{5.109}$$

The initial condition is the error bound in the current time-step:

$$z_x^p(0) = z_e(k) \tag{5.110}$$

Using this error bound, the robust Lipschitz-based MPC is given by

$$\mathbf{u} = \kappa_{mpc}(\mathbf{x}, \hat{\boldsymbol{\theta}}, z_\theta, z_x) = \mathbf{u}^*(0) \tag{5.111a}$$

$$\mathbf{u}^*(.) \triangleq \arg \min_{\mathbf{u}^p_{[0,T]}} J(\mathbf{x}, \hat{\boldsymbol{\theta}}, z_\theta, z_x, \mathbf{u}^p) \tag{5.111b}$$

where

$$J(\mathbf{x}, \hat{\boldsymbol{\theta}}, z_\theta, z_x, \mathbf{u}^p) = \sum_{k=0}^{T-1} L(\mathbf{x}^p(k), \mathbf{u}^p(k)) + W(\mathbf{x}^p(T), z_\theta^p, z_x) \tag{5.112}$$

$s.t.\ \forall k \in [0,\, T]$

$$\mathbf{x}(k+1)^p = \mathbf{x}^p(k) + \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{y}(k)^p, \mathbf{u}(k)^p)\boldsymbol{\theta} + \vartheta(k), \;\; \mathbf{x}(0)^p = \hat{\mathbf{x}}(0) \tag{5.113}$$

$$z_x^p(k+1) = (||\mathbf{A}|| + \mathcal{L}_B \Pi ||\mathbf{H}||) z_x^p(k) + ||\mathbf{B}(\mathbf{y}(k), \mathbf{u}(k))|| z_{\boldsymbol{\theta}}(k) + M_v \tag{5.114}$$

$$z_x^p(0) = z_{x(0)}$$

$$X^p(\tau) \triangleq B(\mathbf{x}^p(k), z_x^p(k)) \subseteq \mathbb{X}, \quad \mathbf{u}^p(k) \in \mathbb{U} \tag{5.115}$$

$$X^p(T) \subseteq \mathbb{X}_f(z_\theta, z_x) \tag{5.116}$$

The effect of the disturbance is built into the uncertainty cone $B(\mathbf{x}^p(k), z_x^p(k))$ via (5.114). Since the uncertainty bound is not monotonically decreasing in this case, the uncertainty radii z_θ and z_x which appears in (5.114) and in the terminal expressions of (5.112) and (5.116) are held constant over the prediction horizon. However, the fact that they are updated at sampling instants when z_θ and z_x shrinks reduces the conservatism of the robust MPC and enlarges the terminal domain that would otherwise have been designed based on a large initial uncertainty $z_\theta(0)$ and $z_x(0)$.

The resulting MPC algorithm is as follows.

Algorithm 8 *The Lipschitz-based MPC algorithm performs as follows:*
At sampling instant k:

1. ***Measure*** *the current output of the plant* $y = y(k)$
2. ***Obtain*** *the current value of the parameter estimates* $\hat{\boldsymbol{\theta}}(k)$ *and the state estimates* $\hat{\mathbf{x}}(k)$ *(Equations* (5.36) *and* (5.3)*) and uncertainty bounds* z_θ *and* z_x *(Equations*(5.48) *and* (5.100)*)*
3. ***Solve*** *the optimization problem* (5.111) *and apply the resulting feedback control law to the plant until the next sampling instant*
4. ***Increment*** $k := k + 1$*; repeat the procedure from step 1 for the next sampling instant.*

5.4 Closed-Loop Robust Stability

By the correct design of the terminal cost W and terminal constraint set X_f, under parameter and state uncertainty, robust stability to the target set Ξ is achieved.

Criterion 3 *The terminal penalty function* $W : \mathbb{X}_f \times \tilde{\Theta}^0 \times \mathcal{X}^0 \to [0, +\infty]$ *and the terminal constraint function* $\mathbb{X}_f : \tilde{\Theta}^0 \times \mathcal{X} \to \mathbb{X}$ *are such that for each* $(\theta,\, \hat{\theta},\, \tilde{\theta}) \in (\Theta^0 \times \Theta^0 \times \tilde{\Theta}^0_\epsilon)$ *and* $(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{e}) \in (\mathcal{X}^0 \times \mathcal{X}^0 \times \mathcal{B}_\epsilon)$ *there exists a feedback* $k_f(\hat{\theta}, \hat{\mathbf{x}}) : \mathbb{X}_f \to \mathbb{U}$ *satisfying*

1. $0 \in \Xi \subseteq \mathbb{X}_f(\tilde{\theta}) \subseteq \mathbb{X}$, $\mathbb{X}_f(\tilde{\theta})$ *closed*
2. $k_f(\hat{\mathbf{x}}, \hat{\theta}) \in \mathbb{U}$, $\forall \mathbf{x} \in \mathbb{X}_f(\tilde{\theta})$
3. $W(\hat{\mathbf{x}}, \tilde{\theta})$ *is continuous with respect to* $\hat{\mathbf{x}} \in \mathbb{R}^{n_x}$
4. $\forall \mathbf{x} \in \mathbb{X}_f(\tilde{\theta}) \backslash \Xi$, $\mathbb{X}_f(\tilde{\theta})$ *is strongly positively invariant under* $k_f(\hat{\mathbf{x}}, \hat{\theta})$ *with respect to* $\hat{\mathbf{x}}_+ \in \mathcal{X} + \mathbf{A}(\hat{\mathbf{x}}, k_f(\hat{\mathbf{x}}, \hat{\theta})) + \mathbf{B}(\mathbf{y}, k_f(\hat{\mathbf{x}}, \hat{\theta}))\Theta + \mathcal{D}$
5. $L(\hat{\mathbf{x}}, k_f(\hat{\mathbf{x}}, \hat{\theta})) + W(\hat{\mathbf{x}}_+, \hat{\theta}) - W(\hat{\mathbf{x}}, \hat{\theta}) \leq 0$, $\forall \mathbf{x} \in \mathbb{X}_f(\tilde{\theta}) \backslash \Xi$.

Condition 5 from criteria 3 requires W to be a local robust Control Lyapunov Function (CLF) for the uncertain system 5.1 with respect to $\theta \in \Theta$, $\vartheta \in \mathcal{D}$ and $\mathbf{x} \in \mathcal{X}$.

Criterion 4 *For any $\tilde{\theta}(1),\ \tilde{\theta}(2) \in \tilde{\Theta}^0$ and $\hat{\mathbf{x}}(1), \hat{\mathbf{x}}(2) \in \mathcal{X}$ s.t. $\left\|\tilde{\theta}(2)\right\| \leq \left\|\tilde{\theta}(1)\right\|$ and $z_x(1) \leq z_x(2)$*

1. $W(\mathbf{x}, \tilde{\theta}(2)) \leq W(\mathbf{x}, \tilde{\theta}(1)),\ \ \forall \mathbf{x} \in \mathbb{X}_f(\tilde{\theta}(1))$
2. $\mathbb{X}_f(\tilde{\theta}(2)) \supseteq \mathbb{X}_f(\tilde{\theta}(1))$

5.4.1 Main Results

Theorem 4 *Let $X_{d0} \triangleq X_{d0}(\Theta^0, \mathcal{X}) \subseteq \mathbb{X}$ denote the set of initial states with uncertainty Θ^0 for which* (5.101) *has a solution. Assuming criteria* 3 *and* 4 *are satisfied, then the closed-loop system, given by (5.1, 5.3, 5.4, 5.8, 5.35, 5.36, 5.48, 5.111), is feasibly asymptotically stabilized from any $x_0 \in X'_{d0}$ to the target set Ξ.*

Proof: The closed-loop stability is established by the feasibility of the control action at each sample time and the strict decrease of the optimal cost J^*. The proof follows from the fact that the control law is optimal with respect to the worst case uncertainty $(\theta,\ \vartheta) \in (\Theta,\ \mathcal{D})$ scenario and the terminal region $\mathbb{X}_f^p$ is strongly positively invariant for 5.1 under the (local) feedback $k_f(.,.)$.

Once the measurement or state estimation is obtained at time-step k, it is possible to obtain the predictions by the model iteration. Using the notation $\mathbf{x}(k+j|k)$ as the prediction at time $k+j$ using the information up to time k. Moreover, the notation $\hat{\mathbf{u}}(k+j|k)$ and $\hat{\mathbf{y}}(k+j|k)$ is used to denote the predictions of the input and output variables. The prediction horizon is denoted by $Hp = T-1$. Considering the full state measurement:

$$
\begin{aligned}
\mathbf{x}^s(k+1|k) &= \tilde{\mathbf{A}}\mathbf{x}_0 + \mathbf{B}(\mathbf{y}_0, \mathbf{u}_0)\boldsymbol{\theta} \\
\mathbf{x}^s(k+2|k) &= \tilde{\mathbf{A}}\mathbf{x}(k+1|k) + \mathbf{B}(\hat{\mathbf{y}}(k+1|k), \hat{\mathbf{u}}(k+1|k))\boldsymbol{\theta} \\
&= \tilde{\mathbf{A}}^2\mathbf{x}_0 + \tilde{\mathbf{A}}\mathbf{B}(\mathbf{y}_0, \mathbf{u}_0)\boldsymbol{\theta} + \mathbf{B}(\hat{\mathbf{y}}(k+1|k), \hat{\mathbf{u}}(k+1|k))\boldsymbol{\theta} \\
\mathbf{x}^s(k+3|k) &= \tilde{\mathbf{A}}^3\mathbf{x}_0 + \tilde{\mathbf{A}}^2\mathbf{B}(\mathbf{y}_0, \mathbf{u}_0)\boldsymbol{\theta} + \tilde{\mathbf{A}}\mathbf{B}(\hat{\mathbf{y}}(k+1|k), \hat{\mathbf{u}}(k+1|k))\boldsymbol{\theta} \\
&\quad + \mathbf{B}(\hat{\mathbf{y}}(k+2|k), \hat{\mathbf{u}}(k+2|k))\boldsymbol{\theta} \\
&\ \ \vdots \\
\mathbf{x}^s(k+Hp|k) &= \tilde{\mathbf{A}}^{Hp}\mathbf{x}_0 + \sum_{i=k, \hat{\mathbf{y}}(k)=y_0, \hat{\mathbf{u}}(k)=u_0}^{Hp-1} \tilde{\mathbf{A}}^k\mathbf{B}(\hat{\mathbf{y}}(i+Hp-1|k), \hat{\mathbf{u}}(i+Hp-1|k))\boldsymbol{\theta}.
\end{aligned}
\tag{5.117}
$$

The same procedure can be repeated for the output feedback. However, the nonlinear part depends on the output only. Thus, the predictions using the state estimates are:

$$\mathbf{x}^o(k+Hp|k) = \tilde{\mathbf{A}}^{Hp}\hat{\mathbf{x}} + \sum_{k=i,\hat{\mathbf{y}}(k)=y_0,\hat{\mathbf{u}}(k)=u_0}^{Hp-1} \tilde{\mathbf{A}}^k \mathbf{B}(\hat{\mathbf{y}}(i+Hp-1|k), \hat{\mathbf{u}}(i+Hp-1|k))\boldsymbol{\theta}. \tag{5.118}$$

The difference between the predictions using the measurement and the state estimation are bounded and can be calculated by:

$$\hat{\mathbf{w}}(k+Hp|k)^p = \mathbf{x}^s(k+Hp|k) - \mathbf{x}^o(k+Hp|k) = \tilde{\mathbf{A}}^{Hp}\mathbf{e}(k) \leq \lambda_{max}(\tilde{\mathbf{A}}^{Hp})z_e \tag{5.119}$$

In the following, the disturbance bound (M_v) is chosen, without loss of generality, to take into account the computable error bound $\lambda_{max}(\tilde{\mathbf{A}}^{Hp})z_e$.

Feasibility: The closed-loop stability is based upon the feasibility of the control action at each sample time. Assuming, at time t, that an optimal solution $\mathbf{u}^p_{[0,T]}$ to the optimization problem (5.101) exists and is found. Let Θ^p and $\mathcal{X}^p$ denote the estimated uncertainty sets at time t and Θ^v, $\mathcal{X}^v$ denote the set at time $t+1$ that would result from the feedback implementation of $\mathbf{u}_t = \mathbf{u}^p_0$. Also, let $\mathbf{x}^p$ represents the worst case state trajectory originating from $\mathbf{x}^p_0 = \mathbf{x}_t$ and $\mathbf{x}^v$ represents the trajectory originating from $\mathbf{x}^v_0 = \mathbf{x} + \mathbf{v}$ for $\mathbf{v} \in \left\{F(\mathbf{x}^a, \mathbf{u}^p) + G(\mathbf{x}^a, \mathbf{u}^p)\Theta^b + \mathcal{D}\right\}$ under the same feasible control input $\mathbf{u}^v_{[1,T]} = \mathbf{u}^p_{[1,T]}$. Moreover, let $X^a_{\Theta^b} \triangleq \{\mathbf{x}^a \,|\, \mathbf{x}^a_+ \in \mathcal{X} + F(\mathbf{x}^a, \mathbf{u}^p) + G(\mathbf{x}^a, \mathbf{u}^p)\Theta^b + \mathcal{D}\}$ which represents the set of all trajectories of the uncertain dynamics.

Since the $\mathbf{u}^p_{[0,T]}$ is optimal with respect to the worst case uncertainty scenario, it suffices to say that $\mathbf{u}^p_{[0,T]}$ drives any trajectory $\mathbf{x}^p \in X^p_{\Theta^p}$ into the terminal region $\mathbb{X}^p_f$. Since the uncertainty for parameter and states are non-expanding over time, we have $\Theta^v \subseteq \Theta^p$ and $\mathcal{X}^v \subseteq \mathcal{X}^p$ implying $\mathbf{x}^v \in X^p_{\Theta^v \times \mathcal{X}^v} \subseteq X^p_{\Theta^p \times \mathcal{X}^p}$. The terminal region $\mathbb{X}^p_f$ is strongly positively invariant for the nonlinear system (5.1) under the feedback $k_f(.,.)$, the input constraint is satisfied in $\mathbb{X}^p_f$ and $\mathbb{X}^v_f \supseteq \mathbb{X}^p_f$ by criteria 3(2.), 3(4.) and 4(2.) respectively. Hence, the input $\mathbf{u} = [\mathbf{u}^p_{[1,T]}, k_{f[T,T+1]}]$ is a feasible solution of (5.101) at time $t+1$ and by induction, the optimization problem is feasible for all $t \geq 0$.

Stability: The stability of the closed-loop system is established by proving strict decrease of the optimal cost $J^*(\hat{\mathbf{x}}, \hat{\theta}, z_\theta, z_x) \triangleq J(\hat{\mathbf{x}}, \hat{\theta}, z_\theta, z_x, \kappa^*)$. First, it is showed that the problem of controlling the system with disturbances and full state measurement is stable. In sequence, it is showed that the uncertainty inserted by the state estimation is a bounded additive disturbance. Let the trajectories $(\mathbf{x}^p,\ \hat{\theta}^p,\ \tilde{\theta}^p, z_\theta^p)$ and control $\mathbf{u}^p$ correspond to any worst case minimizing solution of $J^*(\mathbf{x}, \hat{\theta}, z_\theta, z_x)$. If $\mathbf{x}^p_{[0,T]}$ were extended to $k \in [0, T+1]$ by implementing the feedback $\mathbf{u}^p_{T+1} = k_f(\mathbf{x}^p_{T+1}, \hat{\theta}^p\,)$ then criterion 3(5) guarantees the inequality for state feedback and parameter uncertainty only:

$$L(\mathbf{x}^p_T, k_f(\mathbf{x}^p_T, \hat{\theta}^p_T)\,) + W(\mathbf{x}^p_{T+1}, \tilde{\theta}^p_T) - W(\mathbf{x}^p_T, \tilde{\theta}^p_T) \leq 0. \tag{5.120}$$

The optimal cost

$$\begin{aligned} J^*(\mathbf{x}_t, \hat{\theta}_t, z_{\theta_t}, z_{x_t}) &= \sum_{k=0}^{T-1} L(\mathbf{x}^p_k, \mathbf{u}^p_k) + W(x^p_T, \tilde{\theta}^p_T, z_{xT}) \geq \sum_{k=0}^{T-1} L(\mathbf{x}^p_k, \mathbf{u}^p_k) + W(\mathbf{x}^p_T, \tilde{\theta}^p_T) \\ &\quad + L(x^p_T, k_f(x^p_T, \hat{\theta}^p_T)\,) + W(x^p_{T+1}, \tilde{\theta}^p_T) - W(x^p_T, \tilde{\theta}^p_T) \qquad (5.121) \\ &\geq L(x^p_0, u^p_0) + \sum_{k=1}^{T} L(x^p_k, u^p_k) + L(x^p_T, k_f(x^p_T, \hat{\theta}^p_T)\,) + W(x^p_{T+1}, \tilde{\theta}^p_{T+1}) \\ &\qquad (5.122) \\ &\geq L(x^p_0, u^p_0) + J^*(x_{t+1}, \hat{\theta}_{t+1}, z_{\theta_{t+1}}) \qquad (5.123) \end{aligned}$$

Then, it follows from (5.123) that

$$J^*(x_{t+1}, \hat{\theta}_{t+1}, z_{\theta_{t+1}}) - J^*(x_t, \hat{\theta}_t, z_{\theta_t}) \leq -L(x_t, u_t) \leq -\mu_L(\|x\|). \tag{5.124}$$

where μ_L is a class $\mathcal{K}_\infty$ function. Hence $x(t)$ enters Ξ asymptotically.

■

Theorem 5 *Let* $X'_{d0} \triangleq X'_{d0}(\Theta^0) \subseteq \mathbb{X}$ *denote the set of initial states for which* (5.111) *has a solution. Assuming Assumption* 14 *and Criteria* 3 *and* 4 *are satisfied, then the origin of the closed-loop system given by (5.1, 5.3, 5.4, 5.8, 5.35, 5.36, 5.48, 5.111) is feasibly asymptotically stabilized from any* $x_0 \in X'_{d0}$ *to the target set* Ξ.

The proof of the Lipschitz-based control law follows from that of theorem 4.

5.4.2 Simulation example

Consider the three state reactor from Section 5.2.1. As in GHAFFARI *et al.* (2013), the control objective is to drive the concentrations to a setpoint by manipulating of the product B molar flow rate. The estimated parameters are the same as in Section 5.2.1. The following constraints were used for this simulation example:

$$
\begin{aligned}
-2 \leq u_1 &\leq 2 \\
-0.1 \leq \Delta u_1 &\leq 0.1 \\
-3 \leq x_1 &\leq 3 \\
-3 \leq x_2 &\leq 3 \\
-3 \leq x_3 &\leq 3
\end{aligned} \tag{5.125}
$$

The Lipschitz constraints were added to the problem formulation as described in Subsection 5.3.2 and solved simultaneously with the constraints (5.125).

Two measurements were considered x_1 and x_2, and the third composition x_3 was estimated. The initial condition for the estimator and plant were respectively:

$$
\begin{aligned}
\hat{\mathbf{x}}(0) &= [-0.8, 0.8, 0.4] \\
\mathbf{x}(0) &= [-1, 1, 0.5]
\end{aligned}
$$

The true parameters and their estimates were:

$$
\begin{aligned}
\hat{\boldsymbol{\theta}}(0) &= [0.6, 1.1] \\
\boldsymbol{\theta}(0) &= [0.5, 1]
\end{aligned}
$$

The terminal constraint and the terminal cost were designed using the approach for linear parameter varying (LPV) systems proposed in GAHINET *et al.* (1996) and applied for MPC in ADETOLA *et al.* (2009). For simulation purposes a periodical disturbance was added as a sine function in the measured outputs:

$$
\mathbf{v}(k) = 0.01[\sin(k) \quad \sin(2k)]
$$

Figures 5.15,5.16 and 5.17 show a comparison between the output feedback controller and a state feedback controller, which uses full and perfect plant measurement. The states are recovered by the estimator even in the presence of the disturbance, the output feedback controller is able to drive the system to the origin in the presence of disturbance and parameter uncertainty without any constraint violation. A comparison between the state feedback controller and the output feedback shows an acceptable performance for the output feedback approach. Moreover, the state estimation is improved as the parameters are recovered by the parameter estimation routine as viewed in Figure 5.19, revealing the potential of the adaptive technique. Figure 5.20 shows the uncertainty bound for the auxiliary variable and Figure 5.21 the bound for the parameters uncertainty, in both cases the errors remain bounded during the simulation.

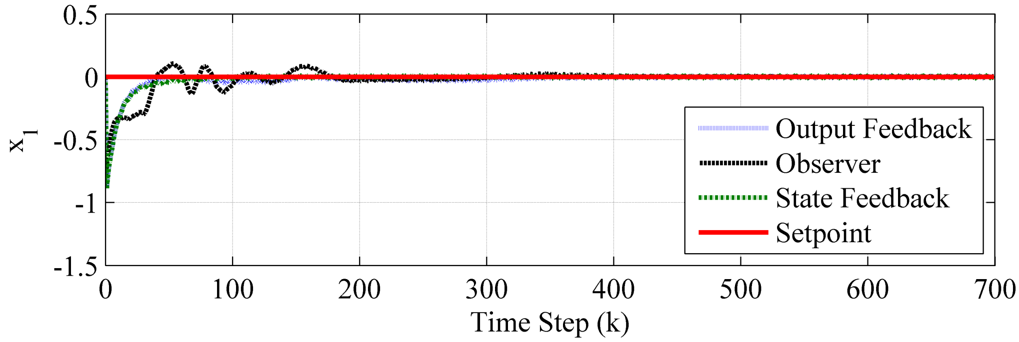

Figure 5.15: x_1 along time for the different controllers.

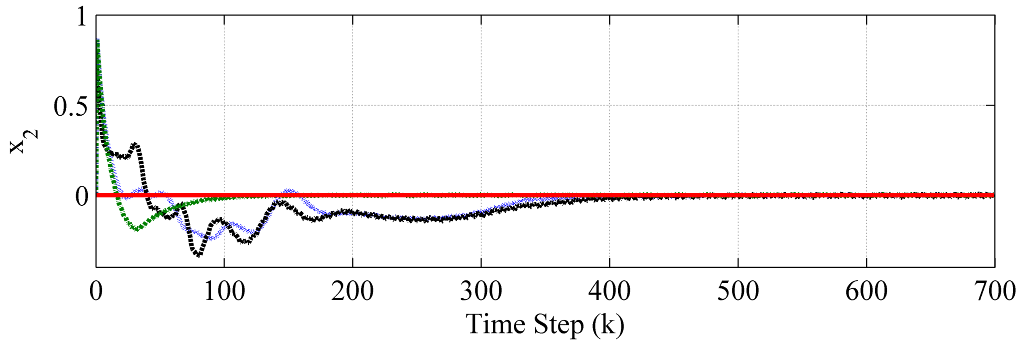

Figure 5.16: x_2 along time for the different controllers.

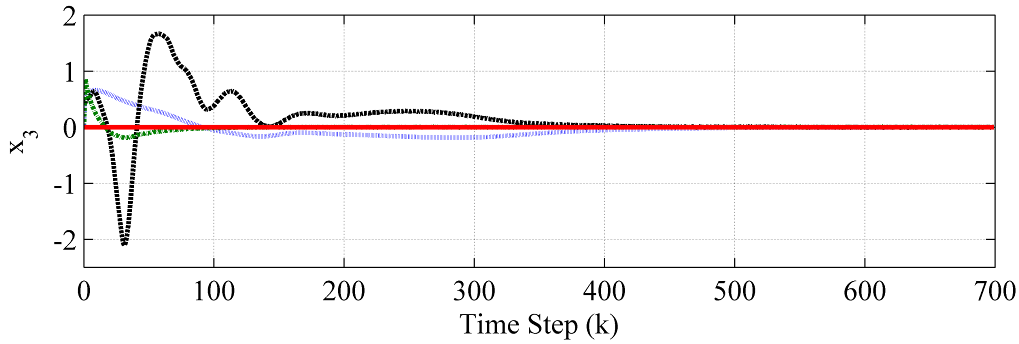

Figure 5.17: x_3 along time for the different controllers.

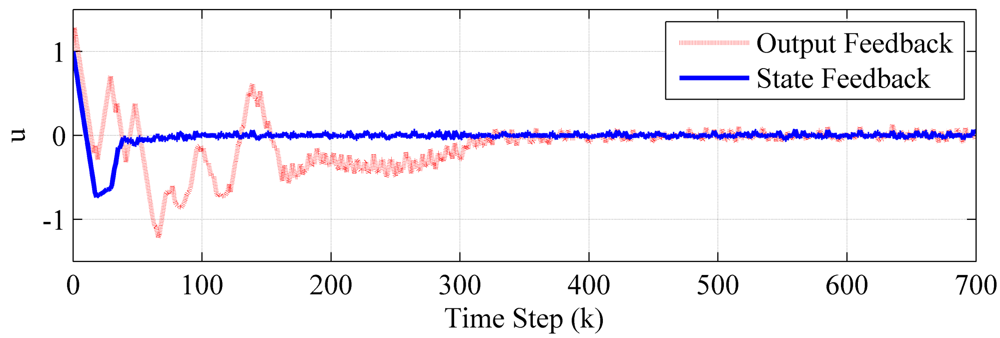

Figure 5.18: Control actions for the output feedback and state feedback controllers.

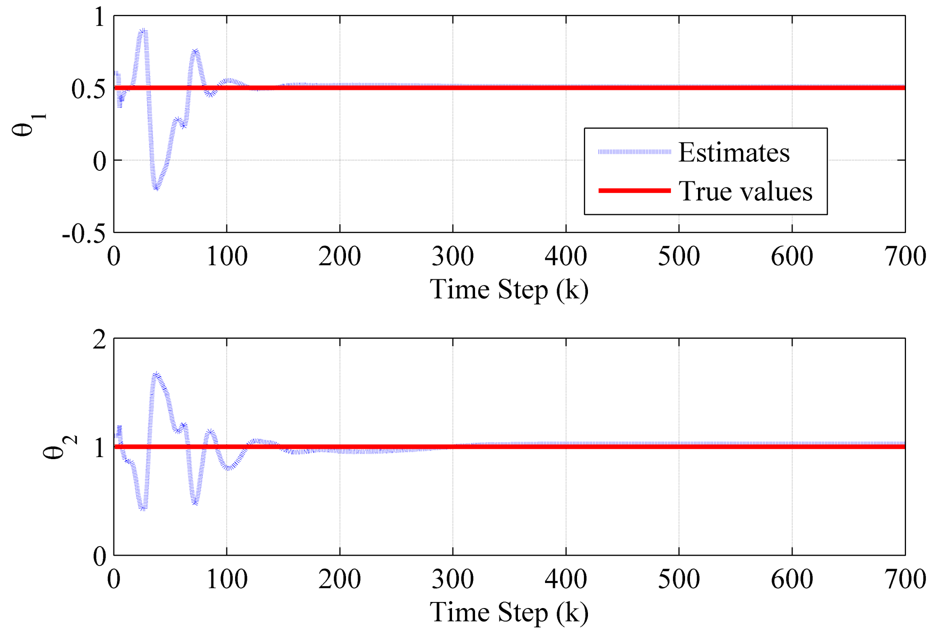

Figure 5.19: Parameters convergence to the true values for the output feedback MPC problem.

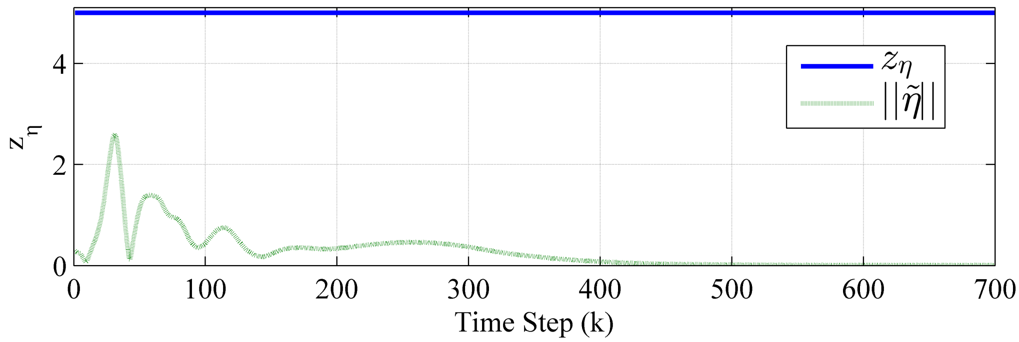

Figure 5.20: Plot of the auxiliary variable uncertainty radius z_{η} and the norm of the auxiliary variable uncertainty along time-step.

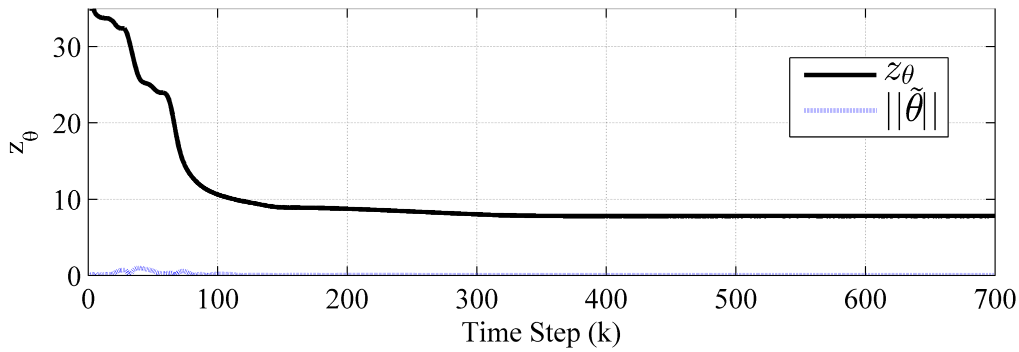

Figure 5.21: Plot of the auxiliary variable uncertainty radius z_{η} and parameter error along time-step.

5.5 Conclusions

A set-based state estimation algorithm for a class of nonlinear discrete-time systems with constant and time-varying parameters was proposed. This estimation routine is able to provide, dynamically, a worst-case uncertainty estimate for the parameters and the state variables. Furthermore, the convergence of the approach was established along with a set update algorithm that guarantees containment of the unknown parameters and state variables. The developed algorithm can deal effectively with systems that are subject to unknown time-varying disturbances. The set-based approach for state and parameter estimation was combined with MPC to provide a controller with robust properties under output feedback that can be solved in real-time by using a Lipschitz-based framework for the uncertainty sets predictions. Simulation results were provided to show the estimation and control performance.

Chapter 6

Final Remarks

6.1 Introduction

The implementation of advanced control techniques, as industrially are called strategies that go beyond regulatory control, has gained prominence since the emergence of the predictive control strategy based on linear models in the 70's. The benefits of using this type of controller range from the economic gain to the process safety improvement. However, the maintenance after the commissioning stage is fundamental, so that the profit obtained in its implementation is sustained. Certainly the controller has its models identified for an operational condition which, in general, can not be maintained throughout the campaign, which can often last for years. The usual solution is to perform new open-loop tests to update the models. Sometimes, the advanced control is turned off, while maintenance is not performed. Following the advanced control trend, the industrial plants instrumentation has gained lots of attention, generating data amounts that are often not used for any purpose other than process monitoring. The main objective of this work was to find a solution to update the models of this type of controller without operational intervention and with guarantees that the stability of the operation is assured.

6.2 Thesis developments

The initial approach focused on the study of state and parameter estimators, since they are used in the NMPC strategy to feedback states even when model parameters are not estimated (Chapter 3). Techniques based on Kalman filtering in their constrained and unconstrained versions were studied. These were selected because of the large use, low computational burden and use of constraints. An effort was made to use the filter in its version *Unscented*, which employs the model simulations directly without the use of linearizations, since sometimes obtaining a linear model, at each sampling instant, in the case of non-linear high-dimensional systems, can be costly. On the other hand, the performance of the classical Kalman filter proved to be superior, especially when the state was augmented to include the parameters to be estimated (Section 3.2). This chapter has shown that a great improvement can be obtained if this widespread algorithms are used for model updating in comparison with the standard additive disturbance approach. Furthermore, a deep study for performance classification in predictive control was showed for the most popular Kalman approaches. Finally, it was showed that the constraint satisfaction can be improved using the adaptive algorithm.

In Chapter 4, an alternative approcah using interval observers was presented in order to update the models used in the predictive controllers and preserve stability. This algorithm was developed for discrete-time systems and can be applied in a large class of systems. Moreover, the computational burden of min-max problem was avoided by using a Lipschitz based method. Finally, the algorithm was successfully tested in the classical van de Vusse reactor and in a medical problem for Chemotherapy dosage in cancer treatment. In this chapter the system states must be measured for application.

In order to overcome the fully state measurement requirement, a state estimator that is able to joint estimate parameters, states and their uncertainty bounds was developed in Chapter 5, Section 5.2. This algorithm is able to estimate constant and dynamic parameters. In this chapter, this interval observer was used for output feedback MPC. A comparison between the state-feedback and the output feedback was carried out for a chemical reactor, showing that, even for partial state measurement, the algorithm was able to recover the true parameters and track the desired setpoint.

6.3 Future research

The state estimation problem still has a lot of open problems. The first one is to develop an algorithm for probabilistic uncertainty estimation in real time for nonlinear systems. A second research topic is the robust constraint satisfaction in presence of structural model uncertainty.

While the linear MPC is a well developed algorithm, the nonlinear algorithm still has open questions. For example, an efficient way for solving the min-max problem is still not available. Moreover, the output feedback problem has no separation principle for a general class of nonlinear systems.

The algorithms in Chapters 4 and 5 can be extended for the economic MPC approach, providing a more lucrative adaptive MPC. Finally, the algorithms must be tested in a laboratory or industrial facility for a final performance validation.

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